MATHEMATICS MAGAZINE

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ROBERT E. HORTON, *Editor*

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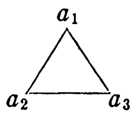
THE TRITRIX, A TRIANGULAR ARRAY OF SCALARS

F. MAX STEIN, Colorado State University

The student in a course in Vector Analysis or Matrix Theory, say, is seldom given the opportunity to inquire as to whether the defined operations of addition and multiplication are the only possible ones. Usually the operation is defined and the discussion progresses without any mention of other possibilities. To show how other definitions may arise, in this paper we shall construct an algebra involving triangular arrays of scalars; later we shall consider the isomorphic systems of vectors and square matrices of a certain type.

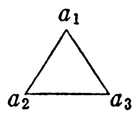
1. Basic Definitions. The *tritrix* is defined as the array

(1)



for which the *elements* a_1, a_2, a_3, called *scalars*, belong to the complex number field [1]. To simplify our notation we write

(2)



$$= (a_1\ a_2\ a_3) = A,$$

so that the scalar at the top of the array corresponds to the first element of the ordered triple, etc.

We make the natural definition of *equality* of two tritrixes to be the equality of corresponding elements; i.e.,

$$A = B \quad \text{if and only if} \quad a_i = b_i, \qquad i = 1, 2, 3, \tag{3}$$

where $B = (b_1\ b_2\ b_3)$.

For a given set of tritrixes, $S = \{A, B, \cdots\}$, we now define the operation of addition, $(+)$, so that S will form a group [1] under this operation. Using the tritrixes A and B as given above, we define the *sum* to be

$$A + B = C, \tag{4}$$

for which $C = (a_1 + b_1\ \ a_2 + b_2\ \ a_3 + b_3)$.

From the definition of addition it readily follows that

(5) a. $A + B = C$, the closure property holds;
b. $A + (B + C) = (A + B) + C$, the operation is associative;
c. $A + 0 = 0 + A = A$ if and only if $0 = (0\ 0\ 0)$, the right and left additive identity;
d. $A + (-A) = (-A) + A = 0$ if and only if $-A = (-a_1\ -a_2\ -a_3)$, the right and left additive inverse; and
e. $A + B = B + A$, the operation is commutative.

Thus tritrixes form an Abelian group under addition with the additive identity member of the set consisting of the scalars zero at each position of the tritrix and the additive inverse of a member of the set consisting of a tritrix with the scalar elements being the negatives of the scalars of the given tritrix.

Up to this point the procedure is analogous to that which one encounters in either a vectors course or a course in matrices. However, when we define the operation of multiplication, $(\cdot)$, we mention several possible choices. If we desire S to form a group under our definition of multiplication, several seemingly likely candidates fail to satisfy one or more of the requirements.

2. Definitions of Multiplication.

DEFINITION I.

$$(a_1\ a_2\ a_3)\cdot(b_1\ b_2\ b_3) = (a_1b_2b_3\ a_2b_3b_1\ a_3b_1b_2). \tag{6}$$

For this definition we see that the closure property is satisfied, and it is easy to show that the associative law fails to hold. A peculiar situation arises in constructing a right identity—it is not unique. To see this we determine scalars x, y, and z such that

$$(a_1\ a_2\ a_3)\cdot(x\ y\ z) = (a_1\ a_2\ a_3). \tag{7}$$

That is, $a_1yz=a_1$, $a_2zx=a_2$, and $a_3xy=a_3$. Since (7) must hold for arbitrary A, we have $yz=1$, $zx=1$, and $xy=1$, or

$$(x\ y\ z) = (1\ 1\ 1) \quad \text{or} \quad (x\ y\ z) = (-1\ -1\ -1). \tag{8}$$

That is, we have two right identities. Furthermore, from (6), it is obvious that neither identity is a left identity for an arbitrary tritrix B.

If we attempt to find a left identity, we get

$$(x\ y\ z)\cdot(a_1\ a_2\ a_3) = (xa_2a_3\ ya_3a_1\ za_1a_2) = (a_1\ a_2\ a_3),$$

or $xa_2a_3=a_1$, $ya_3a_1=a_2$, and $za_1a_2=a_3$, or

$$x = \frac{a_1}{a_2a_3}, \qquad y = \frac{a_2}{a_1a_3}, \qquad z = \frac{a_3}{a_1a_2}, \qquad (\text{if } a_1a_2a_3 \neq 0)$$

which says that the identity would change for each element. Thus there is no unique left identity.

Using the right identities of (8), it is a simple exercise to determine right and left inverses. The final conclusion under definition I is that we are not led to a group.

DEFINITION II.

$$(a_1\ a_2\ a_3)\cdot(b_1\ b_2\ b_3) = (a_1b_1\ a_2b_2\ a_3b_3). \tag{9}$$

Here closure holds, and it can be shown that the associative law holds. The right identity is the same as the left identity which is $I=(1\ 1\ 1)$. For a group we must restrict our tritrixes to arrays for which none of the scalar elements is zero; we then have that

$$\left(\frac{1}{a_1}\ \frac{1}{a_2}\ \frac{1}{a_3}\right)$$

is the inverse of $(a_1\ a_2\ a_3)$. The group thus obtained is obviously Abelian. It is easily shown that $A(B+C)=AB+AC$, and we thus conclude that the set $S=\{0, I, A_1, A_2, \cdots\}$, for which $0=(0\ 0\ 0)$ and $I=(1\ 1\ 1)$, forms a field with the defined operations of addition, of section 1, and multiplication if A_i is of the form $(a_1\ a_2\ a_3)$ . . . provided $a_1a_2a_3\neq 0$.

DEFINITION III.

$$(a_1\ a_2\ a_3)\cdot(b_1\ b_2\ b_3) = a_1b_1 + a_2b_2 + a_3b_3. \tag{10}$$

We dispose of this case immediately since the closure property is not satisfied. Hence, no group results.

DEFINITION IV.

$$(a_1\ a_2\ a_3)\cdot(b_1\ b_2\ b_3) = (a_2b_3 - a_3b_2\ \ a_3b_1 - a_1b_3\ \ a_1b_2 - a_2b_1.) \tag{11}$$

We first see that the closure property holds, and, by proceeding as before, we can conclude that the operation is not associative, there is no identity element, and furthermore the operation is not commutative. Thus, again we do not have a group.

DEFINITION V.

$$(a_1\ a_2\ a_3)\cdot(b_1\ b_2\ b_3) = (a_1b_1\ \ a_2b_1 + a_3b_2\ \ a_3b_3). \tag{12}$$

Here we have closure, the associative law holds, there exists the identity $I=(1\ 0\ 1)$, and each member A of the set possesses an inverse provided $a_1a_3\neq 0$. Thus, this definition leads to a group.

If we strive for an Abelian group under this definition, our set of tritrixes is drastically limited. To see this, for $A\cdot B=B\cdot A$ we get that the first and third elements of the right side of (12) appear properly, but for the second elements we must have

$$a_2b_1 + a_3b_2 = b_2a_1 + a_2b_3,$$

or

$$\frac{a_2}{a_1 - a_3} = \frac{b_2}{b_1 - b_3}$$

for arbitrary members A and B of the set S; we assume that the indicated divisions are permissible. Since these conditions apparently place so many restrictions on the possible members of the set, we do not proceed further along this line.

3. An Examination of the Results. We now look at the various cases in a little different light. Definition I appears to present an acceptable definition of multiplication, but it breaks down at several places if we desire a group. There are two right identities but no left identity, a somewhat unusual situation.

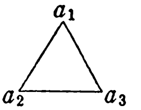
Definition II would appear to be the most logical definition of any to use, since we are led to a field with the seemingly mild restriction imposed that the scalar elements of each tritrix be non-zero except for the additive inverse. However, as we shall see after examining cases III, IV, and V, this is not the definition that lends itself to use in the applications.

Definition III apparently doesn't lead anywhere; however, this is one of the most useful definitions of multiplication. The tritrixes with the operations of addition and multiplication we have defined are isomorphic to vectors with the definition of multiplication being the dot product or scalar product. This is an example which shows that a field is not always the most desirable property to have.

Definition IV also shows a product which leaves us far short of a field. However, the tritrixes with this definition of multiplication are isomorphic to vectors with the cross, or vector, product operation. This is another product which is useful in the applications and which does not lead to a field.

Finally, definition V appears to have been obtained from very little motivation, having some of the features of definition II but not all. Upon closer scrutiny it is observed that the tritrixes of section 1 with the conditions of definition V are isomorphic to second order non-singular matrices with zero in the upper right-hand corner. That is, we can write

(13)



$$= \begin{bmatrix} a_1 & 0 \\ a_2 & a_3 \end{bmatrix}.$$

The extra conditions which are required for the set to form a group under multiplication are not too restrictive, but, if we try to stretch our definition to obtain a field, we apparently ask too much. Actually we have a ring [1] with identity element using this definition of multiplication along with the definition of addition.

Reference

1. Maric J. Weiss, *Higher Algebra for the Undergraduate*, John Wiley, New York, 1949.

Comments on Trickie 57 [November 1962]

Comment by Huseyin Demir. If we introduce the bracket function, we obtain another representation of 31 with four 4's, namely $31 = 4! + [\sqrt{4! + 4! + 4}]$.

Comment by F. Max Stein. Another way to express 31 with four 4's under the given conditions is $31 = 4! + (4! + 4)/4$.

Comment by Alan Sutcliffe. I think **T57** would have been better had factorials not been allowed, for then the alternative solution $\sqrt{4/.\dot{4}} + 4 + 4! = 31$ would not have been possible.

Comment by Harry M. Gehman. The solution suggested is not as good as the one given in the Arithmetic Teacher for October, 1962, p. 308, namely $4! + \sqrt{4} + \sqrt{4}/.4 = 31$.

If we permit the use of the $\sum$ function, where $\sum n = 1 + 2 + \cdots + n$, then there are at least 10 ways of obtaining 31 of which the simplest are $(\sum\sqrt{4})(\sum 4) + 4/4$ and $4\sum 4 - 4/.\dot{4}$.

AN INVESTIGATION OF NINE-DIGIT DETERMINANTS

MARJORIE BICKNELL AND VERNER E. HOGGATT, JR., San Jose State College

1. Introduction. What set of determinant values will occur if the nine positive digits are arranged in a square array and if all possible arrangements are considered is an intriguing question which has been studied by Stancliff [1], Trigg [2], [3], [4], and Hoggatt [5]. Of course, there are $9!/(2\cdot 3!3!)$ or 5040 such distinct determinants which are not equivalent by row exchange, column exchange, or transpose. Directed by Professor Verner E. Hoggatt, Jr., at San Jose State College, Barbara Bodé, Elizabeth O'Connell, and Marjorie Bicknell have established a complete solution.

2. Discussion. A computer program was written to evaluate all distinct determinants. However, certain values did not appear between 0 and the maximum value 412 [5].

Consider all arrangements of the form $S=a\cdot b\cdot c+d\cdot e\cdot f+g\cdot h\cdot i$, for the digits 1 through 9, where no digit is repeated, since two such sums occur in the expansion of the 3×3 determinant,

$$\begin{vmatrix} a & g & f \\ d & b & h \\ i & e & c \end{vmatrix} = (a\cdot b\cdot c + d\cdot e\cdot f + g\cdot h\cdot i) - (b\cdot f\cdot i + h\cdot e\cdot a + c\cdot d\cdot g).$$

Call the sum of the three positive terms P and the sum of the three negative terms N.

By considering all arrangements S by by taking the difference of compatible P and N arrangements, we can show the maximum value obtainable for such determinants, exhibit a determinant of any given possible value, and show whether any specified value occurs by elimination of possibilities of combination. By "compatible," we mean "able to occur in the same determinant." P is compatible with N if and only if any two digits which occur in the same term in P do not occur in the same term in N.

Consider all possible distinct arrangements S. Each term in S is a combination of three digits, so the number of terms which can be formed is $C(9, 3)$ or $9!/(6!3!)$ or 84. Every digit occurs in 84/3 or 28 terms. For each occurrence of a term in an arrangement S, there are three elements fixed in the first term, and $C(6, 3)$ or $6!/(3!3!)$ or 20 ways to choose the second term. The third term consists of the three remaining digits. Hence, for each first term, 20/2 or 10 distinct sums can be formed. Thus there are 28×10 or 280 possible arrangements S available for use as P or N.

3. Supporting Tables. The 280 arrangements S are given in Table I. Examples of determinant values less than 320 are given in Table II in terms of the values of S listed in Table I. Table III contains all determinants greater than or equal to 320, also in terms of the S in Table I.

TABLE I

280 POSSIBLE DISTINCT SUMS IN ORDER OF VALUE

214 =1·8·9+2·5·7+3·4·6	246 =1·6·7+2·8·9+3·4·5	285a=1·3·7+2·8·9+4·5·6
215 =1·7·9+2·5·8+3·4·6	247 =1·7·9+2·4·5+3·6·8	285b=1·7·9+2·3·5+4·6·8
216a=1·8·9+2·5·6+3·4·7	248a=1·4·8+2·5·9+3·6·7	288 =1·3·8+2·6·7+4·5·9
216b=1·8·9+2·6·7+3·4·5	248b=1·4·8+2·7·9+3·5·6	289a=1·3·7+2·6·9+4·5·8
217 =1·7·9+2·4·8+3·5·6	248c=1·8·9+2·3·6+4·5·7	289b=1·3·9+2·5·7+4·6·8
218a=1·6·9+2·5·8+3·4·7	250a=1·4·9+2·5·7+3·6·8	290a=1·4·5+2·7·9+3·6·8
218b=1·7·8+2·4·9+3·5·6	250b=1·5·6+2·7·8+3·4·9	290b=1·4·5+2·8·9+3·6·7
218c=1·7·8+2·5·9+3·4·6	251a=1·4·8+2·6·7+3·5·9	293 =1·4·6+2·5·8+3·7·9
218d=1·8·9+2·4·7+3·5·6	251b=1·5·7+2·4·9+3·6·8	294a=1·4·5+2·7·8+3·6·9
219a=1·7·9+2·5·6+3·4·8	251c=1·5·7+2·8·9+3·4·6	294b=1·4·9+2·3·8+5·6·7
219b=1·7·9+2·6·8+3·4·5	252 =1·5·6+2·7·9+3·4·8	296a=1·4·5+2·6·9+3·7·8
220 =1·6·9+2·5·7+3·4·8	256a=1·4·7+2·6·9+3·5·8	296b=1·4·8+2·3·9+5·6·7
222 =1·6·8+2·5·9+3·4·7	256b=1·6·7+2·3·9+4·5·8	297 =1·3·7+2·6·8+4·5·9
223 =1·6·9+2·4·8+3·5·7	256c=1·6·9+2·3·7+4·5·8	298a=1·5·8+2·3·7+4·6·9
224a=1·7·8+2·5·6+3·4·9	258a=1·5·6+2·8·9+3·4·7	298b=1·6·7+2·4·5+3·8·9
224b=1·7·8+2·6·9+3·4·5	258b=1·5·8+2·4·7+3·6·9	299a=1·5·7+2·3·8+4·6·9
225a=1·5·9+2·6·7+3·4·8	258c=1·7·8+2·4·5+3·6·9	299b=1·5·7+2·4·6+3·8·9
225b=1·5·9+2·6·8+3·4·7	259a=1·3·9+2·7·8+4·5·6	301 =1·3·9+2·4·8+5·6·7
225c=1·6·8+2·4·9+3·5·7	259b=1·4·7+2·6·8+3·5·9	302a=1·2·9+3·6·8+4·5·7
225d=1·8·9+2·4·6+3·5·7	259c=1·7·9+2·3·6+4·5·8	302b=1·3·6+2·8·9+4·5·7
226a=1·6·8+2·5·7+3·4·9	261a=1·5·7+2·4·8+3·6·9	302c=1·5·6+2·4·7+3·8·9
226b=1·6·9+2·7·8+3·4·5	261b=1·5·9+2·3·8+4·6·7	302d=1·7·8+2·3·5+4·6·9
228 =1·6·7+2·5·9+3·4·8	261c=1·5·9+2·4·6+3·7·8	303 =1·3·7+2·5·9+4·6·8
229 =1·5·9+2·7·8+3·4·6	262a=1·4·7+2·5·9+3·6·8	304a=1·2·9+3·6·7+4·5·8
230a=1·6·7+2·5·8+3·4·9	262b=1·4·7+2·8·9+3·5·6	304b=1·3·6+2·7·9+4·5·8
230b=1·6·9+2·4·7+3·5·8	262c=1·5·8+2·3·9+4·6·7	304c=1·4·7+2·5·6+3·8·9
230c =1·7·8+2·3·9+4·5·6	262d=1·6·9+2·4·5+3·7·8	305a=1·4·5+2·6·8+3·7·9
231a=1·7·9+2·3·8+4·5·6	263 =1·3·9+2·6·8+4·5·7	305b=1·5·9+2·3·6+4·7·8
231b=1·7·9+2·4·6+3·5·8	264a=1·4·8+2·5·7+3·6·9	306a=1·2·9+3·5·8+4·6·7
232a=1·5·8+2·6·7+3·4·9	264b=1·4·9+2·5·6+3·7·8	306b=1·2·9+3·7·8+4·5·6
232b=1·5·8+2·6·9+3·4·7	270a=1·3·8+2·7·9+4·5·6	306c=1·3·8+2·4·9+5·6·7
234a=1·6·7+2·4·9+3·5·8	270b=1·4·6+2·7·9+3·5·8	306d=1·8·9+2·3·4+5·6·7
234b=1·6·8+2·7·9+3·4·5	270c=1·4·7+2·5·8+3·6·9	308a=1·5·6+2·3·9+4·7·8
234c=1·8·9+2·3·7+4·5·6	270d=1·5·6+2·4·9+3·7·8	308b=1·6·9+2·3·5+4·7·8
235 =1·5·9+2·4·8+3·6·7	270e=1·6·7+2·3·8+4·5·9	310a=1·3·6+2·7·8+4·5·9
237 =1·4·9+2·6·8+3·5·7	270f =1·6·8+2·3·7+4·5·9	310b=1·3·8+2·5·7+4·6·9
238a=1·4·9+2·7·8+3·5·6	270g=1·8·9+2·3·5+4·6·7	310c=1·4·6+2·5·7+3·8·9
238b=1·5·8+2·4·9+3·6·7	271a=1·3·9+2·6·7+4·5·8	311 =1·3·9+2·5·6+4·7·8
238c=1·5·8+2·7·9+3·4·6	271b=1·4·6+2·7·8+3·5·9	315 =1·2·9+3·5·7+4·6·8
238d=1·8·9+2·4·5+3·6·7	272a=1·3·8+2·6·9+4·5·7	317 =1·3·7+2·5·8+4·6·9
239a=1·5·7+2·6·8+3·4·9	272b=1·7·8+2·3·6+4·5·9	318a=1·2·8+3·6·9+4·5·7
239b=1·5·7+2·6·9+3·4·8	273 =1·4·6+2·8·9+3·5·7	318b=1·4·9+2·3·7+5·6·8
239c=1·6·8+2·4·7+3·5·9	275 =1·3·9+2·5·8+4·6·7	319 =1·2·8+3·5·9+4·6·7
239d=1·7·8+2·4·6+3·5·9	277a=1·5·8+2·4·6+3·7·9	320 =1·4·5+2·6·7+3·8·9
240 =1·4·9+2·6·7+3·5·8	277b=1·6·8+2·4·5+3·7·9	322a=1·2·8+3·6·7+4·5·9
241 =1·6·7+2·4·8+3·5·9	279 =1·5·9+2·3·7+4·6·8	322b=1·4·7+2·3·9+5·6·8
242a=1·4·9+2·5·8+3·6·7	281a=1·4·8+2·5·6+3·7·9	323 =1·3·9+2·4·7+5·6·8
242b=1·6·8+2·3·9+4·5·7	281b=1·5·7+2·3·9+4·6·8	324 =1·2·9+3·4·8+5·6·7
242c=1·6·9+2·3·8+4·5·7	282a=1·3·8+2·5·9+4·6·7	325 =1·2·8+3·7·9+4·5·6
245a=1·4·8+2·6·9+3·5·7	282b=1·4·6+2·5·9+3·7·8	327a=1·3·5+2·8·9+4·6·7
245b=1·5·9+2·4·7+3·6·8	283 =1·5·6+2·4·8+3·7·9	327b=1·7·9+2·3·4+5·6·8

TABLE I (*Continued*)

328 =1·5·8+2·3·6+4·7·9
330a=1·5·6+2·3·8+4·7·9
330b=1·6·8+2·3·5+4·7·9
332a=1·2·9+3·5·6+4·7·8
332b=1·3·6+2·5·9+4·7·8
333a=1·3·5+2·7·9+4·6·8
333b=1·3·7+2·4·9+5·6·8
334 =1·2·8+3·4·9+5·6·7
336a=1·2·7+3·6·9+4·5·8
336b=1·3·8+2·5·6+4·7·9
337 =1·2·8+3·5·7+4·6·9
338 =1·2·7+3·6·8+4·5·9
341 =1·2·7+3·5·9+4·6·8
342 =1·2·9+3·4·7+5·6·8
343 =1·3·5+2·7·8+4·6·9
344 =1·4·8+2·3·7+5·6·9
346 =1·4·7+2·3·8+5·6·9
347 =1·3·5+2·6·9+4·7·8
350a=1·2·7+3·5·8+4·6·9
350b=1·2·7+3·8·9+4·5·6
350c=1·3·6+2·5·8+4·7·9
350d=1·3·8+2·4·7+5·6·9
350e=1·7·8+2·3·4+5·6·9
352 =1·4·9+2·3·6+5·7·8
355a=1·3·7+2·4·8+5·6·9
355b=1·3·9+2·4·6+5·7·8
358a=1·2·8+3·5·6+4·7·9
358b=1·4·6+2·3·9+5·7·8
358c=1·6·9+2·3·4+5·7·8
359 =1·5·7+2·3·6+4·8·9
360a=1·2·6+3·7·8+4·5·9
360b=1·5·6+2·3·7+4·8·9
360c=1·6·7+2·3·5+4·8·9
361 =1·2·6+3·7·9+4·5·8
362 =1·2·7+3·4·9+5·6·8
363 =1·3·5+2·6·8+4·7·9
366 =1·3·4+2·8·9+5·6·7
368 =1·2·6+3·8·9+4·5·7
369 =1·3·7+2·5·6+4·8·9
370a=1·2·8+3·4·7+5·6·9
370b=1·2·9+3·4·6+5·7·8
370c=1·3·6+2·4·9+5·7·8
371 =1·2·6+3·5·9+4·7·8
376 =1·3·6+2·5·7+4·8·9
378 =1·3·4+2·7·9+5·6·8
380 =1·2·7+3·4·8+5·6·9
383 =1·4·8+2·3·6+5·7·9
384 =1·2·6+3·5·8+4·7·9
387a=1·3·5+2·6·7+4·8·9
387b=1·3·8+2·4·6+5·7·9
387c=1·4·6+2·3·8+5·7·9
387d=1·6·8+2·3·4+5·7·9
391 =1·2·5+3·7·9+4·6·8
392 =1·2·7+3·5·6+4·8·9
394a=1·2·5+3·7·8+4·6·9
394b=1·2·5+3·8·9+4·6·7
394c=1·3·4+2·7·8+5·6·9
396 =1·2·5+3·6·9+4·7·8
397 =1·3·6+2·4·8+5·7·9
400a=1·2·6+3·4·9+5·7·8
400b=1·3·4+2·6·9+5·7·8
402 =1·4·9+2·3·5+6·7·8
403a=1·2·8+3·4·6+5·7·9
403b=1·3·9+2·4·5+6·7·8
405a=1·2·6+3·5·7+4·8·9
405b=1·5·9+2·3·4+6·7·8
406 =1·2·5+3·6·8+4·7·9
410 =1·4·5+2·3·9+6·7·8
414 =1·2·9+3·4·5+6·7·8
423a=1·2·6+3·4·8+5·7·9
423b=1·3·4+2·6·8+5·7·9
423c=1·3·5+2·4·9+6·7·8
424a=1·2·5+3·6·7+4·8·9
424b=1·4·7+2·3·6+5·8·9
426a=1·4·6+2·3·7+5·8·9
426b=1·6·7+2·3·4+5·8·9
429 =1·3·7+2·4·6+5·8·9
434a=1·2·4+3·8·9+5·6·7
434b=1·3·6+2·4·7+5·8·9
437 =1·2·4+3·7·9+5·6·8
438 =1·3·4+2·5·9+6·7·8
440 =1·4·8+2·3·5+6·7·9
442a=1·3·8+2·4·5+6·7·9
442b=1·5·8+2·3·4+6·7·9
446a=1·2·4+3·7·8+5·6·9
446b=1·2·7+3·4·6+5·8·9
446c=1·4·5+2·3·8+6·7·9
450 =1·2·4+3·6·9+5·7·8
454a=1·2·5+3·4·9+6·7·8
454b=1·2·8+3·4·5+6·7·9
456a=1·2·6+3·4·7+5·8·9
456b=1·3·4+2·6·7+5·8·9
457 =1·3·5+2·4·8+6·7·9
467 =1·2·4+3·6·8+5·7·9
470 =1·3·4+2·5·8+6·7·9
479 =1·2·4+3·5·9+6·7·8
484 =1·2·5+3·4·8+6·7·9
490 =1·4·7+2·3·5+6·8·9
491 =1·5·7+2·3·4+6·8·9
493 =1·3·7+2·4·5+6·8·9
494a=1·2·4+3·6·7+5·8·9
494b=1·4·5+2·3·7+6·8·9
498 =1·2·3+4·7·9+5·6·8
500 =1·2·3+4·7·8+5·6·9
502 =1·2·3+4·6·9+5·7·8
503 =1·3·5+2·4·7+6·8·9
504 =1·2·3+4·8·9+5·6·7
506a=1·2·4+3·5·8+6·7·9
506b=1·2·7+3·4·5+6·8·9
513 =1·2·3+4·6·8+5·7·9
514 =1·3·4+2·5·7+6·8·9
522 =1·2·3+4·5·9+6·7·8
526 =1·2·5+3·4·7+6·8·9
534 =1·2·3+4·6·7+5·8·9
544 =1·2·3+4·5·8+6·7·9
545 =1·2·4+3·5·7+6·8·9
558a=1·4·6+2·3·5+7·8·9
558b=1·5·6+2·3·4+7·8·9
560 =1·4·5+2·3·6+7·8·9
562 =1·3·6+2·4·5+7·8·9
567 =1·3·5+2·4·6+7·8·9
576a=1·2·6+3·4·5+7·8·9
576b=1·3·4+2·5·6+7·8·9
578 =1·2·3+4·5·7+6·8·9
586 =1·2·5+3·4·6+7·8·9
602 =1·2·4+3·5·6+7·8·9
630 =1·2·3+4·5·6+7·8·9

TABLE II

ONE EXAMPLE OF DETERMINANTS YIELDING EACH VALUE FROM 0 THROUGH 319

0 = 446c − 446b	51 = 270d − 219b	102 = 470 − 368	153 = 514 − 361
1 = 235 − 234b	52 = 277b − 225a	103 = 503 − 400a	154 = 514 − 360a
2 = 220 − 218b	53 = 456a − 403b	104 = 454a − 350d	155 = 491 − 336b
3 = 241 − 238c	54 = 456a − 402	105 = 457 − 352	156 = 526 − 370c
4 = 229 − 225c	55 = 484 − 429	106 = 493 − 387c	157 = 456b − 299a
5 = 234a − 229	56 = 270d − 214	107 = 494b − 387b	158 = 494b − 336b
6 = 446b − 440	57 = 457 − 400a	108 = 454a − 346	159 = 493 − 334
7 = 225a − 218b	58 = 484 − 426a	109 = 506b − 397	160 = 494b − 334
8 = 239c − 231a	59 = 446a − 387a	110 = 494b − 384	161 = 494a − 333a
9 = 232a − 223	60 = 484 − 424b	111 = 470 − 359	162 = 494b − 332a
10 = 456a − 446c	61 = 298b − 237	112 = 330a − 218c	163 = 493 − 330a
11 = 457 − 446b	62 = 456b − 394a	113 = 456a − 343	164 = 494a − 330b
12 = 446c − 434b	63 = 450 − 387a	114 = 484 − 370c	165 = 493 − 328
13 = 252 − 239d	64 = 282b − 218d	115 = 457 − 342	166 = 491 − 325
14 = 456a − 442a	65 = 470 − 405a	116 = 503 − 387c	167 = 456a − 289b
15 = 232a − 217	66 = 442b − 376	117 = 450 − 333a	168 = 526 − 358b
16 = 456a − 440	67 = 490 − 423a	118 = 446a − 328	169 = 503 − 334
17 = 446c − 429	68 = 293 − 225a	119 = 506b − 387b	170 = 506b − 336b
18 = 442a − 424b	69 = 283 − 214	120 = 503 − 383	171 = 526 − 355b
19 = 250b − 231b	70 = 446c − 376	121 = 454b − 333b	172 = 491 − 319
20 = 454b − 434b	71 = 494b − 423a	122 = 493 − 371	173 = 456b − 283
21 = 239b − 218d	72 = 442a − 370b	123 = 506b − 383	174 = 526 − 352
22 = 240 − 218c	73 = 442b − 369	124 = 494b − 370b	175 = 494b − 319
23 = 446b − 423c	74 = 442b − 368	125 = 359 − 234b	176 = 506b − 330a
24 = 262d − 238c	75 = 446c − 371	126 = 484 − 358b	177 = 479 − 302b
25 = 454b − 429	76 = 446c − 370b	127 = 442b − 315	178 = 558a − 380
26 = 277b − 251c	77 = 446c − 369	128 = 434b − 306b	179 = 503 − 324
27 = 252 − 225d	78 = 470 − 392	129 = 526 − 397	180 = 560 − 380
28 = 454b − 426a	79 = 438 − 359	130 = 506a − 376	181 = 503 − 322a
29 = 248a − 219b	80 = 503 − 423a	131 = 491 − 360a	182 = 562 − 380
30 = 454b − 424b	81 = 450 − 369	132 = 484 − 352	183 = 494b − 311
31 = 457 − 426a	82 = 442b − 360a	133 = 503 − 370b	184 = 514 − 330a
32 = 270b − 238d	83 = 446b − 363	134 = 442a − 308a	185 = 493 − 308a
33 = 457 − 424b	84 = 454b − 370c	135 = 493 − 358a	186 = 514 − 328
34 = 304b − 270g	85 = 454b − 369	136 = 506b − 370c	187 = 567 − 380
35 = 251b − 216a	86 = 446a − 360c	137 = 506a − 369	188 = 558a − 370a
36 = 446b − 410	87 = 490 − 403a	138 = 470 − 332a	189 = 514 − 325
37 = 442b − 405a	88 = 438 − 350a	139 = 526 − 387b	190 = 560 − 370a
38 = 277b − 239b	89 = 457 − 368	140 = 467 − 327a	191 = 470 − 279
39 = 261a − 222	90 = 493 − 403a	141 = 493 − 352	192 = 562 − 370a
40 = 440 − 400a	91 = 494b − 403a	142 = 503 − 361	193 = 513 − 320
41 = 446c − 405a	92 = 442b − 350b	143 = 526 − 383	194 = 479 − 285a
42 = 442a − 400a	93 = 493 − 400a	144 = 494b − 305c	195 = 545 − 350c
43 = 446b − 403b	94 = 494b − 400a	145 = 503 − 358b	196 = 558a − 362
44 = 446b − 402	95 = 457 − 362	146 = 506a − 360b	197 = 567 − 370a
45 = 259b − 214	96 = 454b − 358b	147 = 506a − 359	198 = 560 − 362
46 = 456a − 410	97 = 494b − 397	148 = 506b − 358b	199 = 504 − 305a
47 = 270a − 223	98 = 467 − 369	149 = 479 − 330a	200 = 562 − 362
48 = 277b − 229	99 = 470 − 371	150 = 467 − 317	201 = 506b − 305b
49 = 277a − 228	100 = 503 − 403a	151 = 506b − 355b	202 = 500 − 298b
50 = 484 − 434b	101 = 442b − 341	152 = 454b − 302c	203 = 558a − 355a

TABLE II (*Continued*)

204 = 506a − 302b	233 = 567 − 334	262 = 544 − 282b	291 = 522 − 231b
205 = 567 − 362	234 = 558a − 324	263 = 586 − 323	292 = 602 − 310b
206 = 514 − 308a	235 = 576b − 341	264 = 586 − 322b	293 = 545 − 252
207 = 503 − 296b	236 = 586 − 350d	265 = 567 − 302a	294 = 576a − 282a
208 = 558a − 350d	237 = 560 − 323	266 = 576a − 310b	295 = 578 − 283
209 = 545 − 336b	238 = 576b − 338	267 = 526 − 259a	296 = 578 − 282b
210 = 560 − 350a	239 = 576b − 337	268 = 586 − 318b	297 = 586 − 289a
211 = 513 − 302c	240 = 586 − 346	269 = 558a − 289a	298 = 586 − 288
212 = 562 − 350a	241 = 560 − 319	270 = 576a − 306c	299 = 602 − 303
213 = 545 − 332b	242 = 586 − 344	271 = 567 − 296b	300 = 562 − 262c
214 = 504 − 290a	243 = 576a − 333b	272 = 576b − 304a	301 = 578 − 277a
215 = 545 − 330a	244 = 562 − 318b	273 = 576a − 303	302 = 558b − 256a
216 = 562 − 346	245 = 567 − 322a	274 = 576b − 302a	303 = 602 − 299a
217 = 558b − 341	246 = 504 − 258c	275 = 576a − 301	304 = 602 − 298a
218 = 562 − 344	247 = 562 − 315	276 = 558b − 282a	305 = 602 − 297
219 = 560 − 341	248 = 558b − 310b	277 = 576a − 299a	306 = 576a − 270c
220 = 562 − 342	249 = 567 − 318a	278 = 576a − 298a	307 = 578 − 271b
221 = 576a − 355a	250 = 560 − 310b	279 = 514 − 235	308 = 578 − 270d
222 = 558a − 336a	251 = 534 − 283	280 = 586 − 306c	309 = 534 − 225c
223 = 567 − 344	252 = 558a − 306c	281 = 562 − 281b	310 = 558b − 248a
224 = 558a − 334	253 = 586 − 333b	282 = 576a − 294b	311 = 567 − 256b
225 = 567 − 342	254 = 576a − 322b	283 = 562 − 279	312 = 576a − 264a
226 = 576a − 350d	255 = 558b − 303	284 = 545 − 261b	313 = 602 − 289a
227 = 560 − 333b	256 = 562 − 306a	285 = 602 − 317	314 = 602 − 288
228 = 562 − 334	257 = 576b − 319	286 = 558a − 272a	315 = 586 − 271a
229 = 567 − 338	258 = 576a − 318b	287 = 576a − 289b	316 = 586 − 270e
230 = 576a − 346	259 = 576a − 317	288 = 558b − 270c	317 = 578 − 261c
231 = 586 − 355a	260 = 522 − 262b	289 = 586 − 297	318 = 576a − 258b
232 = 576a − 344	261 = 576b − 315	290 = 586 − 296b	319 = 567 − 248a

TABLE III

ALL DETERMINANTS OCCURRING WITH VALUE GREATER THAN OR EQUAL TO 320

320 = 602 − 282a	324: missing	= 586 − 259b	= 545 − 215
= 576b − 256b	325 = 576a − 251b	= 578 − 251a	= 544 − 214
= 576b − 256c	= 576b − 251b	= 545 − 218c	331 = 602 − 271a
= 558a − 238b	= 567 − 242a	328 = 578 − 250b	= 576a − 245b
= 545 − 225a	= 567 − 242b	= 576a − 248a	= 576b − 245b
= 544 − 224a	= 567 − 242c	= 544 − 216a	332 = 602 − 270e
321 = 602 − 281b	= 562 − 237	329: missing	= 602 − 270f
= 560 − 239c	326 = 578 − 252	330 = 602 − 272a	= 560 − 228
= 558b − 237	= 576a − 250a	= 586 − 256a	333 = 558a − 225a
= 545 − 224a	= 560 − 234a	= 586 − 256b	= 558a − 225b
322 = 562 − 240	= 558a − 232a	= 586 − 256c	334 = 576a − 242a
= 544 − 222	= 558a − 232b	= 578 − 248a	= 576b − 242b
323 = 602 − 279	= 545 − 219a	= 578 − 248b	= 576b − 242c
= 586 − 263	= 544 − 218b	= 562 − 232a	= 560 − 226a
= 562 − 239a	= 544 − 218c	= 562 − 232b	335 = 586 − 251a
= 562 − 239b	= 544 − 218d	= 560 − 230a	= 576b − 241
= 558a − 235	327 = 602 − 275	= 560 − 230b	= 560 − 225c

TABLE III (*Continued*)

336 = 578 − 242a	= 602 − 256c	= 586 − 223	386: missing
337 = 578 − 241	= 586 − 240	= 578 − 215	387: missing
= 576b − 239c	= 578 − 232a	364: missing	388 = 630 − 242a
= 567 − 230a	= 576b − 230b	365: missing	389 = 630 − 241
= 562 − 225a	347 = 586 − 239c	366 = 630 − 264a	390 = 630 − 240
= 562 − 225b	= 578 − 231b	367: missing	391 = 630 − 239a
= 560 − 223	= 567 − 220	368 = 630 − 262a	= 630 − 239b
338 = 578 − 240	348 = 578 − 230a	369 = 630 − 261a	= 630 − 239c
= 576a − 238b	349 = 586 − 237	370 = 602 − 232a	392 = 630 − 238b
= 576b − 238b	= 578 − 229	= 602 − 232b	393 = 630 − 237
= 560 − 222	= 567 − 218a	371 = 630 − 259b	394: missing
339 = 602 − 263	350 = 578 − 228	372 = 630 − 258b	395 = 630 − 235
= 578 − 239d	351 = 576b − 225c	= 602 − 230a	396 = 630 − 234a
= 567 − 228	352 = 586 − 234a	373: missing	397: missing
340 = 602 − 262c	353 = 578 − 225a	374 = 630 − 256a	398 = 630 − 232a
= 578 − 238a	= 576b − 223	= 602 − 228	= 630 − 232b
= 578 − 238b	354 = 578 − 224a	375: missing	399: missing
= 578 − 238c	355: missing	376 = 602 − 226a	400 = 630 − 230a
= 560 − 220	356 = 586 − 230b	377 = 602 − 225a	= 630 − 230b
341 = 602 − 261b	357: missing	= 602 − 225b	401: missing
= 586 − 245a	358: missing	378: missing	402 = 630 − 228
= 576a − 235	359 = 578 − 219a	379 = 630 − 251a	403: missing
= 576b − 235	360 = 630 − 270c	= 630 − 251b	404 = 630 − 226a
= 567 − 226a	= 602 − 242b	380 = 630 − 250a	405 = 630 − 225a
342 = 576b − 234a	= 602 − 242c	= 602 − 222	= 630 − 225b
= 560 − 218a	= 578 − 218b	381: missing	= 630 − 225c
343 = 578 − 235	= 578 − 218c	382 = 630 − 248a	406: missing
344 = 586 − 242b	361 = 586 − 225c	= 602 − 220	407 = 630 − 223
= 586 − 242c	= 578 − 217	383: missing	408 = 630 − 222
= 578 − 234a	362: missing	384 = 602 − 218a	409: missing
345 = 586 − 241	363 = 602 − 239a	385 = 630 − 245a	410 = 630 − 220
= 567 − 222	= 602 − 239b	= 630 − 245b	411: missing
346 = 602 − 256b			412 = 630 − 218a

References

1. F. S. Stancliff, "Nine Digit Determinants," Scripta Mathematica, **19** (1953), 278.
2. Charles W. Trigg, "Vanishing Nine Digit Determinants", School Science and Mathematics, **62** (1962), 330–331.
3. Charles W. Trigg, "Unit-Valued Nine-Digit Determinants," Nabla (Bull. Malayan Math. Soc.), **8** (1961), 185–186.
4. C. W. Trigg, American Mathematical Monthly, **69** (1962), 312.
5. Vern Hoggatt and F. W. Saunders, "Maximum Value of a Determinant," American Mathematical Monthly, **62** (1955), 257.

A CENSUS OF NINE-DIGIT DETERMINANTS

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The absolute value of a third order determinant is invariant under interchange of columns, interchange of rows, and interchange of columns and rows. Thus any square array with nine distinct elements is a member of a family of $2(3!)^2$ or 72 arrays, all with determinants having the same absolute value. One-half of the family will be positive and one-half will be negative, unless the determinants of the family vanish. If the elements are the nine positive digits, one member of the family may be designated as the "parent" if, using the notation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$a_1=1$, $a_1<b_1<c_1$, the smallest of the other six digits is in the second row, and $a_2>b_1$, $a_3>b_1$. There are $9!/72$ or 5040 parents.

The parents, in turn, fall naturally into clans of 36 generated from "patriarchs"—parents in which $a_2<b_2<c_2$ and $a_3<b_3<c_3$. Each member of the clan has the same first row. The generation proceeds by setting up a 6 by 6 array and permuting the elements of the second row of the patriarch across the columns and permuting the third row elements down the columns. A typical clan is:

$$\begin{vmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 4 & 6 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 3 & 9 & 8 \\ 4 & 6 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 3 & 9 \\ 4 & 6 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 9 & 3 \\ 4 & 6 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 3 & 8 \\ 4 & 6 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 8 & 3 \\ 4 & 6 & 7 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 4 & 7 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 3 & 9 & 8 \\ 4 & 7 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 3 & 9 \\ 4 & 7 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 9 & 3 \\ 4 & 7 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 3 & 8 \\ 4 & 7 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 8 & 3 \\ 4 & 7 & 6 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 6 & 4 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 3 & 9 & 8 \\ 6 & 4 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 3 & 9 \\ 6 & 4 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 9 & 3 \\ 6 & 4 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 3 & 8 \\ 6 & 4 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 8 & 3 \\ 6 & 4 & 7 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 6 & 7 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 3 & 9 & 8 \\ 6 & 7 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 3 & 9 \\ 6 & 7 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 9 & 3 \\ 6 & 7 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 3 & 8 \\ 6 & 7 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 8 & 3 \\ 6 & 7 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 7 & 4 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 3 & 9 & 8 \\ 7 & 4 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 3 & 9 \\ 7 & 4 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 9 & 3 \\ 7 & 4 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 3 & 8 \\ 7 & 4 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 8 & 3 \\ 7 & 4 & 6 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 7 & 6 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 3 & 9 & 8 \\ 7 & 6 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 3 & 9 \\ 7 & 6 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & 9 & 3 \\ 7 & 6 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 3 & 8 \\ 7 & 6 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 5 \\ 9 & 8 & 3 \\ 7 & 6 & 4 \end{vmatrix}$$

Because of the method of construction of the clan, these determinants are easily reduced to second order determinants by reducing the second and third elements of the first row of each to zero. Only 24 distinct multiplications and subtractions are required. Thus

$$\begin{vmatrix} 2 & -6 \\ -2 & -13 \end{vmatrix} \quad \begin{vmatrix} 3 & -7 \\ -2 & -13 \end{vmatrix} \quad \begin{vmatrix} -13 & -31 \\ -2 & -13 \end{vmatrix} \quad \begin{vmatrix} -7 & -37 \\ -2 & -13 \end{vmatrix} \quad \begin{vmatrix} -15 & -37 \\ -2 & -13 \end{vmatrix} \quad \begin{vmatrix} -10 & -42 \\ -2 & -13 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -6 \\ -1 & -14 \end{vmatrix} \quad \begin{vmatrix} 3 & -7 \\ -1 & -14 \end{vmatrix} \quad \begin{vmatrix} -13 & -31 \\ -1 & -14 \end{vmatrix} \quad \begin{vmatrix} -7 & -37 \\ -1 & -14 \end{vmatrix} \quad \begin{vmatrix} -15 & -37 \\ -1 & -14 \end{vmatrix} \quad \begin{vmatrix} -10 & -42 \\ -1 & -14 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -6 \\ -8 & -23 \end{vmatrix} \quad \begin{vmatrix} 3 & -7 \\ -8 & -23 \end{vmatrix} \quad \begin{vmatrix} -13 & -31 \\ -8 & -23 \end{vmatrix} \quad \begin{vmatrix} -7 & -37 \\ -8 & -23 \end{vmatrix} \quad \begin{vmatrix} -15 & -37 \\ -8 & -23 \end{vmatrix} \quad \begin{vmatrix} -10 & -42 \\ -8 & -23 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -6 \\ -5 & -26 \end{vmatrix} \quad \begin{vmatrix} 3 & -7 \\ -5 & -26 \end{vmatrix} \quad \begin{vmatrix} -13 & -31 \\ -5 & -26 \end{vmatrix} \quad \begin{vmatrix} -7 & -37 \\ -5 & -26 \end{vmatrix} \quad \begin{vmatrix} -15 & -37 \\ -5 & -26 \end{vmatrix} \quad \begin{vmatrix} -10 & -42 \\ -5 & -26 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -6 \\ -10 & -29 \end{vmatrix} \quad \begin{vmatrix} 3 & -7 \\ -10 & -29 \end{vmatrix} \quad \begin{vmatrix} -13 & -31 \\ -10 & -29 \end{vmatrix} \quad \begin{vmatrix} -7 & -37 \\ -10 & -29 \end{vmatrix} \quad \begin{vmatrix} -15 & -37 \\ -10 & -29 \end{vmatrix} \quad \begin{vmatrix} -10 & -42 \\ -10 & -29 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -6 \\ -8 & -31 \end{vmatrix} \quad \begin{vmatrix} 3 & -7 \\ -8 & -31 \end{vmatrix} \quad \begin{vmatrix} -13 & -31 \\ -8 & -31 \end{vmatrix} \quad \begin{vmatrix} -7 & -37 \\ -8 & -31 \end{vmatrix} \quad \begin{vmatrix} -15 & -37 \\ -8 & -31 \end{vmatrix} \quad \begin{vmatrix} -10 & -42 \\ -8 & -31 \end{vmatrix}$$

Finally, the corresponding values of the parents in the clan are:

−38	−53	107	17	121	46
−34	−49	151	61	173	98
−94	−125	51	−135	49	−106
−82	−113	183	−3	205	50
−118	−157	67	−167	65	−130
−110	−149	155	−79	169	−26

The definition requires that there be $(8\cdot 7/2)(6\cdot 5\cdot 4/3!)/2$ or 280 patriarchs, each with its clan of 36. By interchange of columns and rows one patriarch goes into another patriarch, but their clans differ. As a result, each of the 5040 parents appears in one clan, and its reflection about its principal diagonal in another clan. This double appearance provides an automatic computational check.

During odd moments in the summer of 1961, this author evaluated the 5040 parents. The frequency with which the non-negative integers $\leqq 412$ may be represented in absolute value by different *parent* nine-digit determinants is given in the body of the table. The integers represented are shown in conventional fashion on the table perimeter. In every case, the number of nine-digit determinants with absolute value equal to the integer is 72 times the tabular value.

It will be observed that the integer with the greatest number of representations, 48, is 45. The largest representable integer is 412 [1], [2]. There are twenty-four integers with no representation. There are 2880 odd parents and 2160 even parents, so if the nine digits are placed at random in a square array, the probability that the determinant of the array will be even is 3/7 [3]. The 38 parents with zero value [4] and the 32 parents representing unity [5] have been published. Representations of the other integers will be published from time to time.

FREQUENCY OF ABSOLUTE INTEGER VALUES OF PARENT NINE-DIGIT DETERMINANTS

N	0	1	2	3	4	5	6	7	8	9
00	38	32	18	30	14	47	30	25	10	41
01	20	42	32	25	16	36	14	31	39	28
02	35	39	20	22	18	33	19	45	12	21
03	37	26	15	41	25	37	29	27	18	34
04	22	24	23	24	17	48	16	16	18	15
05	25	35	16	21	36	43	11	30	5	18
06	31	17	13	28	11	42	35	24	13	35
07	34	28	22	19	7	28	13	36	31	31
08	26	26	12	24	14	29	6	27	12	15
09	27	25	14	23	13	19	12	22	17	32
10	25	25	25	18	12	27	9	20	22	12
11	29	18	9	21	14	23	15	24	8	18
12	31	22	6	18	17	20	11	14	5	22
13	27	18	19	23	4	30	13	12	20	20
14	11	16	9	21	8	18	11	15	5	15
15	17	14	10	21	14	20	13	14	4	8
16	10	15	17	11	8	32	7	12	8	20
17	8	14	5	7	9	14	9	7	9	9
18	16	12	14	10	10	12	8	13	6	13
19	12	5	13	10	5	17	6	7	16	6
20	9	6	6	9	14	16	4	13	14	22
21	12	7	11	8	5	8	10	7	4	9
22	14	17	11	14	6	15	5	10	7	6
23	5	17	6	6	9	7	10	9	7	7
24	15	8	9	16	8	14	3	7	4	8
25	5	7	12	10	11	16	8	9	8	8
26	6	10	4	11	12	5	9	5	8	5
27	15	6	6	12	9	16	6	7	6	5
28	8	5	7	12	10	10	12	6	4	6
29	3	1	5	3	4	13	5	13	1	4
30	3	3	2	4	3	6	6	6	7	2
31	3	4	1	5	8	10	5	5	4	7
32	6	4	2	5	—	6	9	4	3	—
33	12	3	3	2	4	3	1	6	4	3
34	5	5	2	1	3	2	5	3	1	3
35	1	1	1	2	1	—	1	—	—	1
36	5	2	—	4	—	—	1	—	1	1
37	2	1	2	—	2	—	1	2	—	2
38	2	—	2	—	1	2	—	—	1	1
39	1	3	1	1	—	1	1	—	2	—
40	2	—	1	—	1	3	—	1	1	—
41	1	—	1	—	—	—	—	—	—	—
Totals	616	563	407	542	383	728	389	534	365	513

References

1. Fenton Stancliff, "Nine-Digit Determinants," Scripta Mathematica, **19** (1953), 278.
2. Vern Hoggatt and F. W. Saunders, "Maximum Value of a Determinant," American Mathematical Monthly, **62** (1955), 257.
3. C. W. Trigg and D. C. B. Marsh, "Probability that a Determinant Be Even," American Mathematical Monthly, **70** (1963), 93.
4. Charles W. Trigg, "Vanishing Nine-Digit Determinants," School Science and Mathematics, **62** (1962), 330–331.
5. Charles W. Trigg, "Unit-Valued Nine-Digit Determinants," Nabla (Bull. Malayan Math. Soc.), **8** (1961), 185–186.
6. Charles W. Trigg, "Third Order Determinants Invariant under Element Interchange," The Australian Mathematics Teacher, **18** (1962), 40–41.

Answers

A 313. Write the colors in order diagonally on a cylindrical surface,

$$\begin{matrix} R & W & B & r & w \\ b & R & W & B & r \\ w & b & R & W & B \end{matrix}$$

The lower case letters indicate how the diagonals will fall to complete three vertical columns. By progressing around the cylinder, three distinct patterns are obtained:

RWB	*WBR*	*BRW*
BRW	*RWB*	*WBR*
WBR	*BRW*	*RWB*

The last two could have been obtained from the first by cyclic permutation of the letters. The three clearly are different since the central squares have different colors. All other apparently different arrangements go into one of these *three* by rotation.

A 314. The result is obviously true for two intersecting circles. Consequently it is true for two similar parallel ellipses by orthogonal projection.

A 315. If each side divides the sum of the other three, then each side divides the perimeter P. If no two sides are equal then $P/4 <$ the longest side $< P/2$ so that the longest side equals $P/3$. But then the sum of the four sides is at most $P/3 + P/4 + P/5 + P/6 < P$. Hence at least two sides are equal.

A 316. The factorial of each positive integer, $1 < n < 5$, contains at least one even factor. The factorial of every larger positive integer contains as factors at least two even integers for every integer ending in five. The product of one of these even integers by five introduces a terminating zero. Of the remaining factors, at least one is even, thus insuring that the last non-zero digit of $n!$ is even.

A 317. Obviously the minimum number is n. The maximum number is $n^2 - n + 1$ which occurs in the determinant $|A_{rs}|$ where $A_{rs} = 1 - \delta_{1,r-s}$ where as usual the $\delta_{m,n} = 0$ for $m \neq n$, and $\delta_{m,m} = 1$.

(Quickies on page 206)

FINDING THE Nth ROOT OF A NUMBER BY ITERATION

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The method developed by Newton and Raphson for solving equations by iteration is as follows:

Put the equation in the form $f(x)=0$. Let $y=f(x)$. The equation's roots are now the x-intercepts of the function $y=f(x)$. A guess, x_0, is taken of the intercept. The curve is approximated by a line going through $(x_0, f(x_0))$ with a slope equal to that of the curve at $(x_0, f(x_0))$. The next guess is taken as the x-intercept of the straight line.

In this article we shall discuss the error produced by this method when it is used to find the nth root of a number. If N is the number, the equation to be solved is $x^n=N$, or $x^n-N=0$.

x_0 is our first guess. Then the corresponding straight line (see figure 1) goes through (x_0, x_0^n-N). Its slope is nx_0^{n-1}. The equation of the line is therefore

$$\frac{y - x_0^n + N}{x - x_0} = nx_0^{n-1}.$$

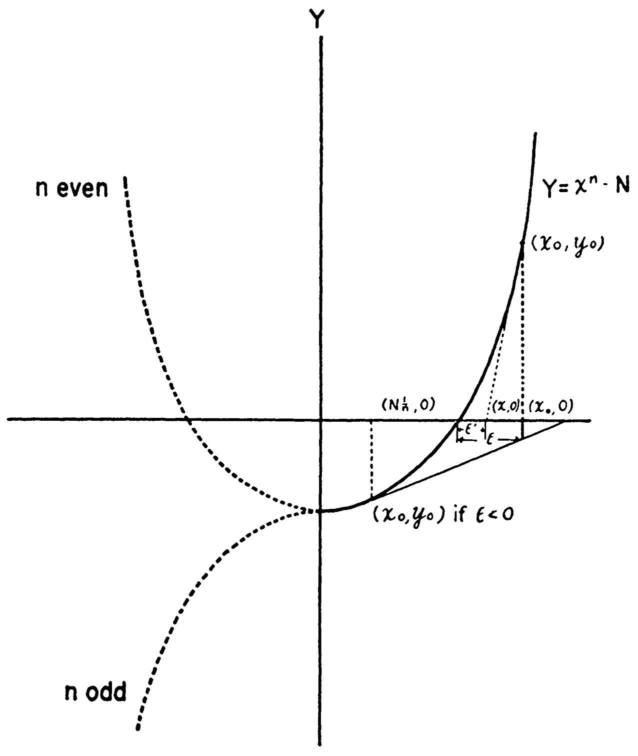

FIG. 1

which has an x-intercept of

$$x = x_0 + \frac{-x_0^n + N}{nx_0^{n-1}} = \frac{(n-1)(x_0^n) + N}{nx_0^{n-1}} = \frac{1}{n}\left[(n-1)x_0 + \frac{N}{x_0^{n-1}}\right].$$

This last form provides the iteration formula. When $n=2$, this becomes

$$x = \frac{1}{2}\left[x_0 + \frac{N}{x_0}\right].$$

This is one familiar way to take the square root of a number. We average the first guess and the quotient obtained by dividing the number by the guess.

How rapidly do we find the square root? Let $x_0 = \sqrt{N}+\epsilon$, where ϵ is our error term. The size of ϵ tells us how accurate our guess is. Then the quotient is:

$$\begin{array}{r|l} & \sqrt{N} - \epsilon + \dfrac{\epsilon^2}{\sqrt{N}+\epsilon} \\ \hline \sqrt{N}+\epsilon) & N \\ & \underline{N + \epsilon\sqrt{N}} \\ & -\epsilon\sqrt{N} \\ & \underline{-\epsilon\sqrt{N} - \epsilon^2} \\ & \qquad\qquad \epsilon^2 \end{array}$$

Our next guess is obtained by averaging the quotient and the divisor, our first guess. This is

$$\sqrt{N} + \frac{\epsilon^2}{2\sqrt{N}+2\epsilon}.$$

Since our first guess was a positive number, $\sqrt{N}+\epsilon$ is greater than zero. Therefore, even if ϵ is negative our new error is positive. If our original error was positive, our new error is less than both $\epsilon/2$ and $\epsilon^2/2\sqrt{N}$. Thus after our second guess the error must decrease and approach zero as the number of trials approaches infinity. The beauty of this method is the fact that the error term decreases as the *square* of the old error term. If N is greater than $\frac{1}{4}$, and our old guess is accurate to four decimal places, our new one is accurate to at least seven, almost twice as many!

We shall now consider the general case. The guess of $N^{1/n}+\epsilon$, using similar notation, is raised to the $n-1$st power and divided into N. We expand $(N^{1/n}+\epsilon)^{n-1}$ by the binomial theorem.

$$\begin{aligned}(N^{1/n} + \epsilon)^{n-1} &= N^{(n-1)/n} + (n-1)\epsilon N^{(n-2)/n} + \cdots + \binom{n-1}{k}\epsilon^k N^{(n-k-1)/n} + \cdots + e^{n-1} \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k} e^k N^{(n-k-1)/n}\end{aligned}$$

Dividing, using sigma notation:

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\epsilon^k N^{(n-k-1)/n}\,\overline{\big)N}^{\,N^{1/n}-(n-1)\epsilon}$$

$$\frac{N+\sum_{k=1}^{n-1}\binom{n-1}{k}e^k N^{(n-k)/n}}{}$$

$$-(n-1)\epsilon N^{(n-1)/n}-\sum_{k=2}^{n-1}\binom{n-1}{k}\epsilon^k N^{(n-k)/n}$$

$$\underline{-(n-1)\epsilon N^{(n-1)/n}-\sum_{k=1}^{n-1}(n-1)\binom{n-1}{k}\epsilon^{k+1}N^{(n-k-1)/n}}$$

$$\sum_{k=1}^{n-1}(n-1)\binom{n-1}{k}\epsilon^{k+1}N^{(n-k-1)/n}-\sum_{k=2}^{n-1}\binom{n-1}{k}\epsilon^k N^{(n-k)/n}$$

This last expression is the remainder. We shall put it in a more manageable form. Shifting the "dummy" in the first expression:

$$R=\sum_{k=2}^{n}(n-1)\binom{n-1}{k-1}\epsilon^k N^{(n-k)/n}-\sum_{k=2}^{n-1}\binom{n-1}{k}\epsilon^k N^{(n-k)/n}$$

$$=(n-1)\epsilon^n+\sum_{k=2}^{n-1}\epsilon^k N^{(n-k)/n}\left[(n-1)\binom{n-1}{k-1}-\binom{n-1}{k}\right].$$

Examining the expression in the brackets,

$$(n-1)\binom{n-1}{k-1}-\binom{n-1}{k}=\frac{(n-1)(n-1)!}{(k-1)!(n-k)!}-\frac{(n-1)!}{k!(n-k-1)!}$$

$$=\frac{(n-1)!}{(k-1)!(n-k-1)!}\left[\frac{n-1}{n-k}-\frac{1}{k}\right]=\frac{(n-1)!}{(k-1)!(n-k-1)!}\left[\frac{nk-n}{k(n-k)}\right]$$

$$=\frac{n(k-1)}{k}\binom{n-1}{k-1}.$$

Hence

$$R=(n-1)\epsilon^n+\sum_{k=2}^{n-1}\epsilon^k N^{(n-k)/n}\left[\frac{n(k-1)}{k}\binom{n-1}{k-1}\right].$$

The quotient is

$$\frac{N}{x_0^{n-1}}=N^{1/n}-(n-1)\epsilon+\frac{R}{(N^{1/n}+\epsilon)^{n-1}}.$$

This gives

$$x=N^{1/n}+\frac{R}{n(N^{1/n}+\epsilon)^{n-1}}.$$

We observe once again that the term containing ϵ drops out. Our new error, ϵ', is equal to

$$\frac{R}{n(N^{1/n}+\epsilon)^{n-1}}.$$

Even if ϵ was negative, ϵ' is positive; $y=x^n-N$ slopes away more sharply than the straight line that is approximating it (see diagram). We shall consider only cases in which ϵ is positive because this must be the case for all guesses after the first.

We wish to find the size of the error.

$$(n-1)\epsilon^n = \epsilon^n N^{(n-n)/n}\frac{n(n-1)}{n}\binom{n-1}{n-1}$$

so R equals

$$\sum_{k=2}^{n}\epsilon^k N^{(n-k)/n}\frac{n(k-1)}{k}\binom{n-1}{k-1} = n\sum_{k=2}^{n}\epsilon^k N^{(n-k)/n}\frac{k-1}{k}\binom{n-1}{k-1}.$$

ϵ' is then

$$\frac{\sum_{k=2}^{n}\epsilon^k N^{(n-k)/n}\frac{k-1}{k}\binom{n-1}{k-1}}{(N^{1/n}+\epsilon)^{n-1}}.$$

This is less than both

$$\frac{n-1}{n}\epsilon \quad\text{and}\quad \frac{(n-1)\epsilon^2}{2N^{1/n}}.$$

We shall compare the numerator with the denominator as expanded by the binomial theorem.

$$(N^{1/n}+\epsilon)^{n-1} = \sum_{k=0}^{n-1}N^{(n-k-1)/n}\epsilon^k\binom{n-1}{k} = \sum_{j=1}^{n}N^{(n-j)/n}\epsilon^{j-1}\binom{n-1}{j-1}.$$

Therefore:

$$\epsilon' = \frac{1}{(N^{1/n}+\epsilon)^{n-1}}\sum_{k=2}^{n}\frac{\epsilon^k N^{(n-k)/n}(k-1)\binom{n-1}{k-1}}{k}$$
$$< \frac{1}{(N^{1/n}+\epsilon)^{n-1}}\,\frac{\epsilon(n-1)}{n}\sum_{k=2}^{n}\epsilon^{k-1}N^{(n-k)/n}\binom{n-1}{k-1}.$$

But the sigma expansion of $(N^{1/n}+\epsilon)^{n-1}$ is the same as the above expansion except that it has one more term. Hence,

$$\epsilon' < \frac{\epsilon(n-1)}{n}.$$

It remains to prove the other inequality,

$$\epsilon' < \frac{(n-1)\epsilon^2}{2N^{1/n}}.$$

$$\epsilon' = \frac{\epsilon^2}{(N^{1/n}+\epsilon)^{n-1}} \sum_{k=2}^{n} \frac{\epsilon^{k-2}N^{(n-k)/n}(k-1)}{k}\binom{n-1}{k-1}$$

$$= \frac{\epsilon^2}{(N^{1/n}+\epsilon)^{n-1}} \sum_{j=0}^{n-2} \frac{\epsilon^{j}N^{(n-j-2)/n}(j+1)}{j+2}\binom{n-1}{j+1}$$

$$= \frac{(n-1)\epsilon^2}{2(N^{1/n}+\epsilon)^{n-1}} \sum_{k=0}^{n-2} \frac{\epsilon^{k}N^{(n-k-2)/n}(k+1)2}{(k+2)(n-1)}\binom{n-1}{k+1}$$

$(N^{1/n}+\epsilon)^{n-2}$ is equal to, by the binomial expansion,

$$\sum_{k=0}^{n-2} \epsilon^k N^{(n-k-2)/n}\binom{n-2}{k}.$$

If it is shown that each term in this expansion for $(N^{1/n}+\epsilon)^{n-2}$ is greater than or equal to the corresponding term in the other sigma expansion we will have shown that

$$\epsilon' < \frac{(n-1)\epsilon^2}{2(N^{1/n}+\epsilon)}.$$

Therefore we ask:

$$\binom{n-2}{k} \overset{?}{\geqq} \frac{2(k+1)}{(n-1)(k+2)}\binom{n-1}{k+1}.$$

Expanding both sides:

$$\frac{(n-2)(n-3)\cdots(n-k-1)}{(1)(2)\cdots(k)} \overset{?}{\geqq} \frac{2(k+1)(n-1)(n-2)\cdots(n-k-1)}{(k+2)(1)(2)\cdots(k+1)(n-1)}$$

or, canceling,

$$1 \overset{?}{\geqq} \frac{2}{k+2}.$$

This is true, for k varies from 0 to $n-2$. Therefore

$$\epsilon' < \frac{(n-1)\epsilon^2}{2(N^{1/n}+\epsilon)} < \frac{(n-1)\epsilon^2}{2N^{1/n}}.$$

Thus, after the second guess, each succeeding approximation is closer to the nth root of N. The error approaches 0 as the number of approximations approach infinity. If ϵ is large compared to $N^{1/n}$, ϵ' is close to $(n-1)\epsilon/n$. Thus, if n is large, ϵ may approach 0 slowly. Once ϵ is small compared to $N^{1/n}$, ϵ' is about $(n-1)\epsilon^2/2N^{1/n}$. The number of decimal places to which our guess is accurate is approximately doubled with each trial.

In general, if one desires the nth root of a number, a good approximation should first be obtained by other methods. It is only when ϵ is small that Newton's method can be used profitably.

The following illustration will demonstrate the effectiveness of this method in taking the cube root of 2.

$\sqrt[3]{2}=1.259921\cdots$ This figure will be useful in obtaining our error term. Since we are taking a cube root, to obtain a new guess we square the old one and divide it into 2. Then a weighted average is taken, the old guess counting twice as much as the quotient.

As our first guess try 1. ϵ_1 is negative. $2/1^2=2$. The weighted average is $\frac{1}{3}(2(1)+2)=1.333\cdots$ $\epsilon_2=0.07$. The error term is now positive. Again using the iteration formula $\sqrt[3]{2}=1.264$, $\epsilon_3=0.004$. We notice that $\epsilon_3<\epsilon_2^2$. Since ϵ_2 is small compared to $\sqrt[3]{2}$, the formula for ϵ_3 can be approximated by $\epsilon_3=\epsilon_2^2/\sqrt[3]{2}$. If greater accuracy is desired one more trial yields $\sqrt[3]{2}=1.259934$. $\epsilon_4=0.000013$.

I would like to express my gratitude to Dr. I. A. Dodes, chairman of the mathematics department of the Bronx High School of Science for the help and encouragement which he gave me in writing this article.

REFLECTIVE GEOMETRY OF THE BROCARD POINTS

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INTRODUCTION

The circle described on and having for diameter the segment joining the circumcenter and symmedian point of triangle ABC is known as the Brocard circle of the triangle. The perpendicular bisectors of sides BC, CA, AB pass through the circumcenter and meet the Brocard circle again at respective points A_1, B_1, C_1. Triangle $A_1B_1C_1$ is called Brocard's first triangle. A_1B, B_1C, C_1A pass through the positive Brocard point of triangle ABC while A_1C, B_1A, C_1B are concurrent at the negative Brocard point of the given triangle. Triangles ABC and $A_1B_1C_1$ are inversely similar and consequently lines through A, B, C parallel respectively to B_1C_1, C_1A_1, A_1B_1 meet at a point on the circumcircle of triangle ABC (Steiner point). This result is a special case of a more general theorem which may be applied to any pair of inversely similar triangles lying in the plane. Lines through A_1, B_1, C_1 parallel to BC, CA, AB meet at K, the symmedian point of triangle ABC, which lies on circle $A_1B_1C_1$. It is easy to demonstrate that point K has a position in triangle $A_1B_1C_1$ homologous to the position of the Steiner point in triangle ABC. Accordingly, K becomes the Steiner point of the first Brocard triangle.

A contemplation of this parallel construction whereby homologous Steiner points in inversely similar triangles ABC and $A_1B_1C_1$ are obtained suggested to the author the possibility of an extension of this process. Would it be possible to construct parallels through the vertices of triangle $ABC(A_1B_1C_1)$ to the sides of triangle $A_1B_1C_1(ABC)$ in more than one way to obtain concurrent rays and possibly homologous points in the two triangles? If so, what of the nature of these additional points? Some of the results of this investigation are listed in the theorems that follow.

We first record a couple of theorems used extensively in obtaining the conclusions to be presently enumerated. Let rays from the vertices of triangle ABC through point P determine cevian triangle DEF (Fig. 1). Considering triangle ADC and transversal BPE, we apply the theorem of Menelaus and have

$$\frac{DB}{BC}\cdot\frac{CE}{EA}\cdot\frac{AP}{PD} = -1 \quad \text{or} \quad \frac{BD}{BC} = \frac{DP}{PA}\cdot\frac{AE}{EC}.$$

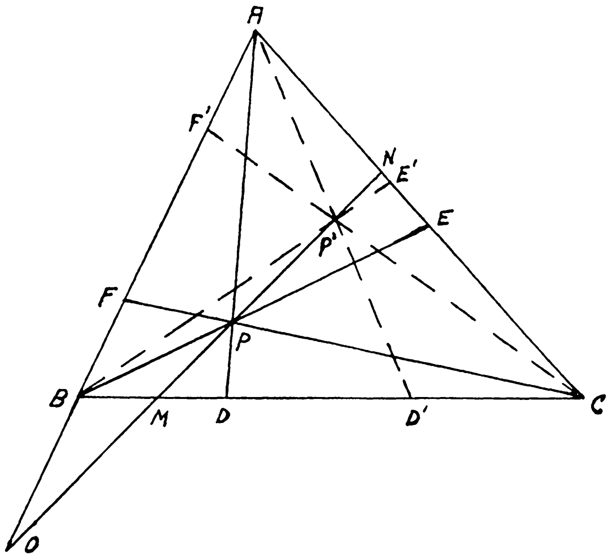

Fig. 1

Turning to triangle ABD and transversal FPC, we obtain

$$\frac{BC}{CD}\cdot\frac{DP}{PA}\cdot\frac{AF}{FB} = -1 \quad \text{or} \quad \frac{DC}{BC} = \frac{DP}{PA}\cdot\frac{AF}{FB}.$$

Addition then yields

$$\frac{BD}{BC} + \frac{DC}{BC} = \frac{DP}{PA}\left(\frac{AF}{FB} + \frac{AE}{EC}\right)$$

which becomes

$$1 = \frac{DP}{PA}\left(\frac{AF}{FB} + \frac{AE}{EC}\right) \quad \text{or} \quad \frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}.$$

Similar relationships may be derived with respect to vertices B and C after which the well known equalities follow:

Theorem 1. *Rays from the vertices of triangle ABC through point P determine points D, E, F on sides BC, CA, AB respectively so that*

$$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}, \qquad \frac{BP}{PE} = \frac{BD}{DC} + \frac{BF}{FA}, \qquad \frac{CP}{PF} = \frac{CE}{EA} + \frac{CD}{DB}.$$

Let P' be another point in the plane of triangle ABC having cevian triangle $D'E'F'$ (Fig. 1). Construct line PP' to meet BC, CA, AB at respective points M, N, O. Then

$$\frac{BM}{MC}=\frac{\dfrac{AE}{EC}-\dfrac{AE'}{E'C}}{\dfrac{AF}{FB}-\dfrac{AF'}{F'B}}$$

and similar expressions exist for CN/NA and AO/OB. These values will not be derived here since considerable space is needed for their verification. They are obtained through the use of material recorded in the article entitled "A Triangle Theorem" occurring in the April, 1960 issue of *School Science and Mathematics*. The author has used them for many years and has found them to be of immense service in investigating properties of the triangle.

THEOREM 2. *Rays from the vertices of triangle ABC through points P and P' determine points D, E, F and D', E', F' on sides BC, CA, AB respectively. A line through P and P' meets BC, CA, AB at respective points M, N, O. Then*

$$\frac{BM}{MC}=\frac{\dfrac{AE}{EC}-\dfrac{AE'}{E'C}}{\dfrac{AF}{FB}-\dfrac{AF'}{F'B}},\qquad \frac{CN}{NA}=\frac{\dfrac{BF}{FA}-\dfrac{BF'}{F'A}}{\dfrac{BD}{DC}-\dfrac{BD'}{D'C}},\qquad \frac{AO}{OB}=\frac{\dfrac{CD}{DB}-\dfrac{CD'}{D'B}}{\dfrac{CE}{EA}-\dfrac{CE'}{E'A}}.$$

It should be mentioned that the results given in the two preceding theorems are always true as long as the segments involved are considered as directed quantities. That is, if D lies between B and C, then BD/DC is considered positive. If D lies without B and C, on BC extended, BD/DC is to be considered negative. Similarly for the other segments involved in the results given.

Some ratio values also needed in future computations are now listed. Their derivation may be found in texts dealing with the geometry of the triangle. The sides opposite vertices A, B, C of triangle ABC are represented by a, b, c.

(a) If P is the positive Brocard point, then

$$\frac{BD}{DC}=\frac{c^2}{a^2},\qquad \frac{CE}{EA}=\frac{a^2}{b^2},\qquad \frac{AF}{FB}=\frac{b^2}{c^2}.$$

(b) When P is the negative Brocard point

$$\frac{BD}{DC}=\frac{a^2}{b^2},\qquad \frac{CE}{EA}=\frac{b^2}{c^2},\qquad \frac{AF}{FB}=\frac{c^2}{a^2}.$$

(c) For the isotomic conjugate of the positive Brocard point

$$\frac{BD}{DC}=\frac{a^2}{c^2},\qquad \frac{CE}{EA}=\frac{b^2}{a^2},\qquad \frac{AF}{FB}=\frac{c^2}{b^2}.$$

(d) For the isotomic conjugate of the negative Brocard point

$$\frac{BD}{DC}=\frac{b^2}{a^2},\qquad \frac{CE}{EA}=\frac{c^2}{b^2},\qquad \frac{AF}{FB}=\frac{a^2}{c^2}.$$

Since vertex A_1 of Brocard's first triangle is the point of intersection of rays from B and C through the positive and negative Brocard points respectively, we easily determine

(e) If P is vertex A_1 of Brocard's first triangle, then

$$\frac{BD}{DC}=\frac{b^2}{c^2},\qquad \frac{CE}{EA}=\frac{a^2}{b^2},\qquad \frac{AF}{FB}=\frac{c^2}{a^2}.$$

In similar fashion we have

(f) When P is vertex B_1 of Brocard's first triangle

$$\frac{BD}{DC}=\frac{a^2}{b^2},\qquad \frac{CE}{EA}=\frac{c^2}{a^2},\qquad \frac{AF}{FB}=\frac{b^2}{c^2}.$$

(g) If P is vertex C_1 of Brocard's first triangle, then

$$\frac{BD}{DC}=\frac{c^2}{a^2},\qquad \frac{CE}{EA}=\frac{b^2}{c^2},\qquad \frac{AF}{FB}=\frac{a^2}{b^2}.$$

Let us now utilize the tools that have been provided. Suppose that AD, BE, CF be rays through A_1, a vertex of Brocard's first triangle. Using Theorem 1, we have

$$\frac{CA_1}{A_1F}=\frac{CE}{EA}+\frac{CD}{DB}$$

which becomes

$$\frac{CA_1}{A_1F}=\frac{a^2+c^2}{b^2}$$

when the ratios given in (e) are used. Let AD', BE', CF' be rays through P', the isotomic conjugate of the negative Brocard point. Then

$$\frac{CP'}{P'F'}=\frac{CE'}{E'A}+\frac{CD'}{D'B}=\frac{a^2+c^2}{b^2}$$

after the ratios of (d) are substituted therein. As

$$\frac{CA_1}{A_1F}=\frac{CP'}{P'F'},$$

it becomes evident that A_1P' is parallel to side AB of triangle ABC.

In identical fashion we may show that B_1P' is parallel to BC, while C_1P' is parallel to CA. Accordingly, this result:

THEOREM 3. *Through vertices A_1, B_1, C_1 of Brocard's first triangle rays are* *drawn parallel respectively to sides AB, BC, CA of triangle ABC. These rays are concurrent at the isotomic conjugate of the negative Brocard point of triangle ABC.*

Using the same methods and the ratios listed in (c), (e), (f), (g) we have

THEOREM 4. *Through vertices A_1, B_1, C_1 of Brocard's first triangle rays are drawn parallel respectively to sides CA, AB, BC of triangle ABC. These rays are concurrent at the isotomic conjugate of the positive Brocard point of triangle ABC.*

It is thus demonstrated that there are three ways in which parallels may be constructed to the sides of triangle ABC through the vertices of triangle $A_1B_1C_1$ to obtain concurrent rays. As we have seen these parallels meet at the symmedian point, the isotomic conjugate of the negative Brocard point, and the isotomic conjugate of the positive Brocard point of triangle ABC.

We immediately wonder as to whether there are two additional ways to construct parallels to the sides of triangle $A_1B_1C_1$ through the vertices of triangle ABC to secure concurrent rays. A little thought reveals such to be the case and furthermore the two points thus obtained will have positions in triangle ABC homologous to the positions that the points of Theorems 3 and 4 have with respect to triangle $A_1B_1C_1$.

Let us demonstrate the truth of this statement by using a more elaborate method than is actually required. This is done in order that some important ratio values associated with these points may be secured. For this purpose construct through C_1 and A_1, vertices of Brocard's first triangle, a line to meet sides BC and AB of triangle ABC at points M and O (Fig. 2). Let AA_1, BA_1, CA_1 meet sides BC, CA, AB at points D, E, F and allow AC_1, BC_1, CC_1 to meet sides BC, CA, AB at points D', E', F'. The use of Theorem 2 gives

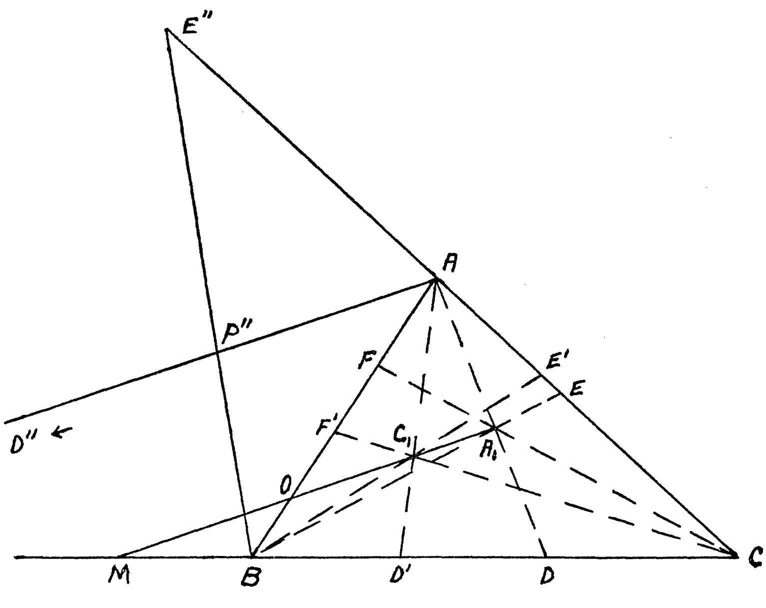

FIG. 2

$$\frac{AO}{OB} = \frac{\dfrac{CD}{DB} - \dfrac{CD'}{D'B}}{\dfrac{CE}{EA} - \dfrac{CE'}{E'A}} = \frac{c^4 - a^2b^2}{a^2c^2 - b^4}$$

when (e) and (g) are used. Also

$$\frac{BM}{MC} = \frac{\dfrac{AE}{EC} - \dfrac{AE'}{E'C}}{\dfrac{AF}{FB} - \dfrac{AF'}{F'B}} = \frac{b^4 - a^2c^2}{b^2c^2 - a^4}$$

through the continued use of (e) and (g).

We may write

$$\frac{AO}{OB} = \frac{c^4 - a^2b^2}{a^2c^2 - b^4} \quad \text{as} \quad \frac{c - OB}{OB} = \frac{c^4 - a^2b^2}{a^2c^2 - b^4}$$

and solve for OB thereby obtaining

$$OB = \frac{c(a^2c^2 - b^4)}{(c^2 - b^2)(a^2 + b^2 + c^2)} \cdot$$

In like manner the equation

$$\frac{BM}{MC} = \frac{b^4 - a^2c^2}{b^2c^2 - a^4}$$

may be written as

$$\frac{BM}{MB + a} = \frac{b^4 - a^2c^2}{b^2c^2 - a^4} \cdot$$

Solving this equation and remembering that $MB = -BM$, we obtain

$$BM = \frac{a(b^4 - a^2c^2)}{(b^2 - a^2)(a^2 + b^2 + c^2)} \cdot$$

Through A construct a parallel to C_1A_1 meeting BC at D'' (Fig. 2). Triangles $AD''B$ and OMB are then similar and

$$\frac{BD''}{BM} = \frac{c}{OB} \cdot$$

Replacing OB and BM by the values just announced for them, we find that

$$BD'' = \frac{a(b^2 - c^2)}{(b^2 - a^2)} \cdot \quad \text{Then} \quad \frac{BD''}{D''C} = \frac{BD''}{D''B + a}$$

and treating BD'' as a directed segment we eventually determine that

$$\frac{BD''}{D''C} = \frac{b^2 - c^2}{c^2 - a^2}.$$

If a parallel to A_1B_1 through B meet CA at E'', we proceed in the same way to show that

$$\frac{CE''}{E''A} = \frac{c^2 - a^2}{a^2 - b^2}.$$

Finally, a parallel to B_1C_1 through C meets AB at F'' so that

$$\frac{AF''}{F''B} = \frac{a^2 - b^2}{b^2 - c^2}.$$

Forming the product

$$\frac{BD''}{D''C} \cdot \frac{CE''}{E''A} \cdot \frac{AF''}{F''B}$$

and substituting the values just determined therein, it is found that this product equals positive unity. The converse of Ceva's theorem then tells us that rays AD'', BE'', CF'' are concurrent at a point P''.

Now AD'' and BE'' are parallel respectively to C_1A_1 and A_1B_1. Remembering that triangles ABC and $A_1B_1C_1$ are inversely similar, we see that $\angle AP''B = 180° - \angle A$. As BE'' and CF'' are parallel to A_1B_1 and B_1C_1, $\angle BP''C = \angle B$. Finally, CF'' and AD'' are parallel to B_1C_1 and C_1A_1 which shows that $\angle CP''A = \angle C$.

Suppose that Ω' be the negative Brocard point of triangle ABC (Fig. 3).

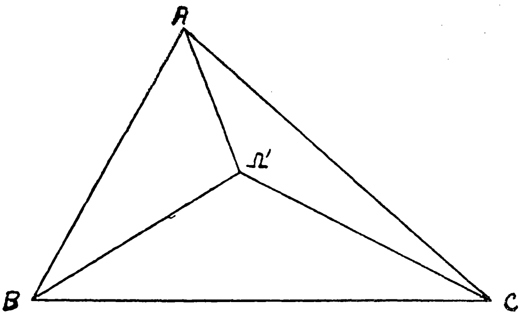

FIG. 3

We experience no difficulty in showing that $\angle A\Omega'B = 180° - \angle A$, $\angle B\Omega'C = 180° - \angle B$, and $\angle C\Omega'A = 180° - \angle C$. Comparing the angles in Figures 2 and 3 we observe that $\angle A\Omega'B = \angle AP''B$, $\angle B\Omega'C$ is supplementary to $\angle BP''C$, and $\angle C\Omega'A$ is supplementary to $\angle CP''A$. So the angles formed by rays from Ω' to the vertices of triangle ABC are either equal or supplementary to the corresponding angles formed by rays from P'' to the vertices of triangle ABC. This is sufficient to show that Ω' and P'' are a pair of reflective points with respect to triangle ABC (D. Moody Bailey, "Some Reflective Geometry of the Triangle," MATHEMATICS MAGAZINE, May–June, 1960).

Accordingly, if circles $B\Omega'C$, $C\Omega'A$, $A\Omega'B$ of the indirect group of adjoint circles associated with the Brocard configuration be reflected about sides BC, CA, AB respectively, then the reflected circles will pass through point P''. Let us now designate points P'', D'', E'', F'' of Fig. 2 as points P', D', E', F' and summarize our findings in this manner:

THEOREM 5. *Through A, B, C rays are drawn parallel respectively to sides C_1A_1, A_1B_1, B_1C_1 of Brocard's first triangle. These rays are concurrent at point P' and meet sides BC, CA, AB at points D', E', F' so that*

$$\frac{BD'}{D'C}=\frac{b^2-c^2}{c^2-a^2},\qquad \frac{CE'}{E'A}=\frac{c^2-a^2}{a^2-b^2},\qquad \frac{AF'}{F'B}=\frac{a^2-b^2}{b^2-c^2}.$$

The negative Brocard point and P' are a pair of reflective points with respect to triangle ABC.

A chain of similar reasoning establishes the truth of the following statement:

THEOREM 6. *Through A, B, C rays are drawn parallel respectively to sides A_1B_1, B_1C_1, C_1A_1 of Brocard's first triangle. These rays are concurrent at point P and meet sides BC, CA, AB at points D, E, F so that*

$$\frac{BD}{DC}=\frac{a^2-b^2}{b^2-c^2},\qquad \frac{CE}{EA}=\frac{b^2-c^2}{c^2-a^2},\qquad \frac{AF}{FB}=\frac{c^2-a^2}{a^2-b^2}.$$

The positive Brocard point and P are a pair of reflective points with respect to triangle ABC.

A reexamination of the angles made at P and P' by rays from these points to the vertices of triangle ABC reveals, as previously mentioned, that points P' and P of Theorems 5 and 6 have the same position with respect to triangle ABC that the isotomic conjugates of the Brocard points of triangle ABC have with respect to triangle $A_1B_1C_1$. The isotomic conjugate of the positive Brocard point of triangle ABC is the reflective partner of the negative Brocard point of triangle $A_1B_1C_1$ while the isotomic conjugate of the negative Brocard point of triangle ABC is the reflective partner of the positive Brocard point of triangle $A_1B_1C_1$.

By extending side B_1C_1 of Brocard's first triangle to meet BC at M and AB at O and proceeding as we did prior to the statement of Theorem 5, we may determine the ratios associated with the Steiner point. This gives

THEOREM 7. *Through A, B, C rays are drawn parallel respectively to sides B_1C_1, C_1A_1, A_1B_1 of Brocard's first triangle. These rays are concurrent at the Steiner point S and meet sides BC, CA, AB at points D, E, F so that*

$$\frac{BD}{DC}=\frac{c^2-a^2}{a^2-b^2},\qquad \frac{CE}{EA}=\frac{a^2-b^2}{b^2-c^2},\qquad \frac{AF}{FB}=\frac{b^2-c^2}{c^2-a^2}.$$

Having obtained ratio values for the rays passing through P and P', the reflective partners of the Brocard points, we next construct line PP' to meet

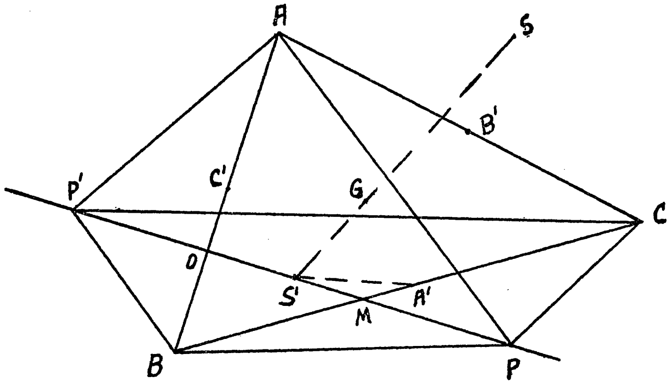

FIG. 4

sides BC, CA, AB of triangle ABC at respective points M, N, O (Fig. 4). Theorem 2 yields

$$\frac{BM}{MC} = \frac{\dfrac{AE}{EC} - \dfrac{AE'}{E'C}}{\dfrac{AF}{FB} - \dfrac{AF'}{F'B}} \quad \text{or} \quad \frac{BM}{MC} = \frac{b^2 - a^2}{c^2 - a^2}$$

when the ratio values given in Theorems 5 and 6 are substituted. We similarly determine CN/NA and AO/OB to get

THEOREM 8. *P and P' are the reflective partners of the positive and negative Brocard points respectively of triangle ABC. Line PP' meets sides BC, CA, AB at respective points M, N, O so that*

$$\frac{BM}{MC} = \frac{b^2 - a^2}{c^2 - a^2}, \quad \frac{CN}{NA} = \frac{c^2 - b^2}{a^2 - b^2}, \quad \frac{AO}{OB} = \frac{a^2 - c^2}{b^2 - c^2}.$$

When any straight line meets sides BC, CA, AB of triangle ABC at points M, N, O we may allow M', N', O' to be the harmonic conjugates of these points with respect to the ends of the corresponding sides of triangle ABC. AM', BN', CO' may then be shown to be concurrent at a point Q'. Line MNO is said to be the trilinear polar of Q' and Q' is called the pole of line MNO.

Let M', N', O' be the harmonic conjugates of points M, N, O of Theorem 8. We then write

$$\frac{BM'}{M'C} = \frac{a^2 - b^2}{c^2 - a^2}, \quad \frac{CN'}{N'A} = \frac{b^2 - c^2}{a^2 - b^2}, \quad \frac{AO'}{O'B} = \frac{c^2 - a^2}{b^2 - c^2}.$$

Rechecking the ratios listed in Theorem 7 leads to this conclusion:

THEOREM 9. *Q' is the isotomic conjugate of the Steiner point of a given triangle. The trilinear polar of Q' passes through the reflective partners of the Brocard points of the triangle.*

Through the use of Theorem 2 we may show that a line through the centroid and symmedian point of triangle ABC meets sides BC, CA, AB at points M, N, O so that

$$\frac{BM}{MC} = \frac{c^2 - a^2}{b^2 - a^2}, \qquad \frac{CN}{NA} = \frac{a^2 - b^2}{c^2 - b^2}, \qquad \frac{AO}{OB} = \frac{b^2 - c^2}{a^2 - c^2}.$$

This fact together with the ratios of Theorems 7 and 8 enables us to state:

THEOREM 10. *The trilinear polar of the Steiner point of triangle ABC contains the centroid and symmedian point of the triangle and meets sides BC, CA, AB at points M', N', O'. If M, N, O be selected on sides BC, CA, AB isotomic to points M', N', O' then line MNO passes through the reflective partners of the Brocard points of the triangle.*

Utilizing Theorem 2 we find another interesting property of line MNO.

THEOREM 11. *Line PP', through the reflective partners of the Brocard points of triangle ABC, meets sides BC, CA, AB at respective points M, N, O. Rays AM'', BN'', CO'' are constructed isogonal to respective rays AM, BN, CO. Line $M''N''O''$ contains the circumcenter and symmedian point of triangle ABC.*

It is possible to deal with P and P', together with their isotomic conjugates, and list many relationships comparable to those expressed in Theorems 9, 10, and 11. However, we shall be content to list one other of this nature.

THEOREM 12. *The trilinear polar of P (P'), the reflective partner of the positive (negative) Brocard point of triangle ABC, passes through the centroid and the isotomic conjugate of the positive (negative) Brocard point of triangle ABC.*

It is thus seen from Theorems 10 and 12 that the trilinear polars of the three points P, S, P' are concurrent at the common centroid of triangles ABC and $A_1B_1C_1$. This happens to be sufficient to show that the six points A, B, C, P, S, P' lie on a conic. Also the trilinear polar of the Steiner point of triangle ABC passes through the Steiner point of triangle $A_1B_1C_1$ while the trilinear polar of the reflective partner of the positive (negative) Brocard point of triangle ABC passes through the reflective partner of the negative (positive) Brocard point of triangle $A_1B_1C_1$.

Suppose a line be passed through vertex A_1 of Brocard's first triangle and the centroid of triangle ABC to meet side BC at point M. The use of Theorem 2, together with the ratios of (e), and the fact that rays through the centroid give ratios $BD/DC=1$, $CE/EA=1$, and $AF/FB=1$ enables us to determine that

$$\frac{BM}{MC} = \frac{b^2 - a^2}{c^2 - a^2}.$$

A line through B_1 and the centroid meets CA at N so that

$$\frac{CN}{NA} = \frac{c^2 - b^2}{a^2 - b^2}$$

while a line through C_1 and the centroid meets AB at O so that

$$\frac{AO}{OB} = \frac{a^2 - c^2}{b^2 - c^2}.$$

Theorem 8 shows that points M, N, O are on line PP'. It is well known that triangles ABC and $A_1B_1C_1$ have a common centroid and that Brocard's first and second triangles, both lying on the Brocard circle, are in perspective at this centroid. This gives

THEOREM 13. *Lines connecting the corresponding vertices of Brocard's first and second triangles are in perspective at the centroid of triangle ABC and these lines meet BC, CA, AB at points M, N, O respectively. Points M, N, O determine a line passing through the reflective partners of the Brocard points of triangle ABC.*

We return now to Fig. 4 and note that rays AP' and CP are each parallel to side C_1A_1 of Brocard's first triangle. Likewise, rays BP' and AP are parallel to side A_1B_1 while rays CP' and BP are each parallel to side B_1C_1. Let $A'B'C'$ be the medial triangle of triangle ABC and through A', B', C' construct parallels to B_1C_1, C_1A_1, A_1B_1 respectively. These three rays are easily seen to be concurrent at point S', the midpoint of PP', since A', B', C' are the midpoints of BC, CA, AB. As these rays are parallel to the sides of $A_1B_1C_1$, they must be parallel to the corresponding sides of Brocard's first triangle associated with triangle $A'B'C'$. This means that these three rays are concurrent at the Steiner point of triangle $A'B'C'$ which is known to be on the circumcircle of triangle $A'B'C'$ or on the nine point circle of triangle ABC. Consequently S', the midpoint of PP' lies on the nine point circle of triangle ABC. We then have

THEOREM 14. *P and P' are the reflective partners of the Brocard points of triangle ABC. S', the midpoint of PP', lies on the nine point circle of triangle ABC and is the Steiner point of medial triangle $A'B'C'$.*

We know that the centroid G of triangle ABC is the center of similitude of triangles ABC and $A'B'C'$. As S and S' are corresponding points in these triangles, we may construct segment SS' which must pass through G and be trisected there. As S' has been shown to be the midpoint of PP', we now see that G is the centroid of triangle PSP'. Similitude then shows that the midpoint of SP' is the reflective partner of the positive Brocard point of triangle $A'B'C'$ while the midpoint of PS is the reflective partner of the negative Brocard point of triangle $A'B'C'$.

THEOREM 15. *P and P' are the reflective partners of the positive and negative Brocard points respectively of triangle ABC and S is the Steiner point of this triangle. The midpoints of SP' and PS are the reflective partners of the positive and negative Brocard points respectively of medial triangle $A'B'C'$. Triangles ABC and PSP' have a common centroid G.*

Recalling that the symmedian point, the isotomic conjugate of the negative

Brocard point, the isotomic conjugate of the positive Brocard point, and the centroid G of triangle ABC have positions with respect to triangle $A_1B_1C_1$ homologous to the positions of points S, P, P', and G with respect to triangle ABC, we may interpret Theorem 15 in this way:

THEOREM 16. *The triangle having for vertices the symmedian point, and the isotomic conjugates of the Brocard points of triangle ABC has the same centroid as triangle ABC and the midpoint of the segment joining the isotomic conjugates of the Brocard points lies on the nine point circle of triangle $A_1B_1C_1$.*

It is known and recorded in texts on triangle geometry that the triangle formed by the Brocard points and the isotomic conjugate of the symmedian point has its centroid at G. Theorem 16 shows that the triangle formed by the isotomic conjugates of these three points with respect to triangle ABC also has its centroid at G. Through an extended computation involving the use of Theorem 2, the following similar conclusion is attained:

THEOREM 17. *The isotomic conjugates with respect to triangle ABC of vertices A_1, B_1, C_1 of Brocard's first triangle are determined. Triangle ABC and the triangle formed by these isotomic points have a common centroid.*

We can determine ratio values for the isotomic conjugates of points P', P, and S by inverting each of the ratios recorded in Theorems 5, 6, and 7. When this is done it is not difficult to demonstrate the truth of the following statement:

THEOREM 18. *P and P' are the reflective partners of the positive and negative Brocard points respectively of triangle ABC and S is the Steiner point of this triangle. The isotomic conjugates with respect to triangle ABC of points P, S, P' lie on the line at infinity.*

This means that the three rays to each of the points from the vertices of triangle ABC are parallel.

The midpoint of the segment joining any pair of reflective points in a triangle is known to lie on the nine point circle of the triangle. This fact is demonstrated in the article previously mentioned dealing with reflective geometry. Using this fact and referring to Theorem 14, we have

THEOREM 19. *P and P' are the reflective partners of Ω and Ω' respectively, the positive and negative Brocard points of triangle ABC. The midpoints of segments ΩP, $\Omega' P'$, PP' determine the nine point circle of triangle ABC.*

Let rays through P from the vertices of triangle ABC meet sides BC, CA, AB at respective points D, E, F. Suppose that P' be a point in medial triangle $A'B'C'$ homologous to the position of P in triangle ABC. If rays from vertices A, B, C through P' determine points D', E', F' on sides BC, CA, AB it can be shown that

$$\frac{BD'}{D'C}=\frac{\dfrac{AF}{FB}+1}{\dfrac{AE}{EC}+1},\qquad \frac{CE'}{E'A}=\frac{\dfrac{BD}{DC}+1}{\dfrac{BF}{FA}+1},\qquad \frac{AF'}{F'B}=\frac{\dfrac{CE}{EA}+1}{\dfrac{CD}{DB}+1}.$$

Using these formulaes and the ratios given in Theorems 5, 6, and 7 we obtain

THEOREM 20. *P' is a point in triangle ABC having ratio values*

$$\frac{BD'}{D'C}=\left(\frac{a^2-c^2}{b^2-c^2}\right)^2,\qquad \frac{CE'}{E'A}=\left(\frac{b^2-a^2}{c^2-a^2}\right)^2,\qquad \frac{AF'}{F'B}=\left(\frac{c^2-b^2}{a^2-b^2}\right)^2.$$

P' is the reflective partner of the negative Brocard point of triangle $A'B'C'$.

THEOREM 21. *P' is a point in triangle ABC having ratio values*

$$\frac{BD'}{D'C}=\left(\frac{c^2-b^2}{a^2-b^2}\right)^2,\qquad \frac{CE'}{E'A}=\left(\frac{a^2-c^2}{b^2-c^2}\right)^2,\qquad \frac{AF'}{F'B}=\left(\frac{b^2-a^2}{c^2-a^2}\right)^2.$$

P' is the reflective partner of the positive Brocard point of triangle $A'B'C'$.

THEOREM 22. *P' is a point in triangle ABC having ratio values*

$$\frac{BD'}{D'C}=\left(\frac{b^2-a^2}{c^2-a^2}\right)^2,\qquad \frac{CE'}{E'A}=\left(\frac{c^2-b^2}{a^2-b^2}\right)^2,\qquad \frac{AF'}{F'B}=\left(\frac{a^2-c^2}{b^2-c^2}\right)^2.$$

P' is the Steiner point of triangle $A'B'C'$.

It is demonstrated in the article "Some Reflective Geometry of the Triangle" that the isogonal conjugates of a pair of reflective points with respect to triangle ABC is a pair of inverse points with respect to circle ABC. It is also known that

$$\frac{BD}{DC}\cdot\frac{BD'}{D'C}=\frac{c^2}{b^2},\qquad \frac{CE}{EA}\cdot\frac{CE'}{E'A}=\frac{a^2}{c^2},\qquad \frac{AF}{FB}\cdot\frac{AF'}{F'B}=\frac{b^2}{a^2}$$

where P and P' are isogonal conjugates in triangle ABC. Making use of these truths and the ratios given in Theorems 5 and 6, we secure these worthwhile facts:

THEOREM 23. *Ω is the positive Brocard point of triangle ABC with Ω and Q being inverse points with respect to circle ABC. If rays from the vertices of triangle ABC through point Q determine cevian triangle DEF, then*

$$\frac{BD}{DC}=\frac{c^2}{b^2}\left(\frac{c^2-a^2}{b^2-c^2}\right),\qquad \frac{CE}{EA}=\frac{a^2}{c^2}\left(\frac{a^2-b^2}{c^2-a^2}\right),\qquad \frac{AF}{FB}=\frac{b^2}{a^2}\left(\frac{b^2-c^2}{a^2-b^2}\right).$$

THEOREM 24. *Ω' is the negative Brocard point of triangle ABC with Ω' and Q' being inverse points with respect to circle ABC. If rays from the vertices of triangle ABC through point Q' determine cevian triangle $D'E'F'$, then*

$$\frac{BD'}{D'C} = \frac{c^2}{b^2}\left(\frac{b^2 - c^2}{a^2 - b^2}\right), \quad \frac{CE'}{E'A} = \frac{a^2}{c^2}\left(\frac{c^2 - a^2}{b^2 - c^2}\right), \quad \frac{AF'}{F'B} = \frac{b^2}{a^2}\left(\frac{a^2 - b^2}{c^2 - a^2}\right).$$

The points of Theorems 23 and 24 are known to lie on the Lemoine axis of triangle ABC.

As a final offering we list a relationship involving any point P that lies on the nine point circle of a given triangle. This relationship has been recently derived by the author and the proof, being rather extended, is not given here. The result may be used to demonstrate that point S' of Theorems 14 and 22 lies on the nine point circle of triangle ABC. It will also show that the points of Theorems 20 and 21 do not lie on this circle.

THEOREM 25. *If P is any point on the nine point circle of triangle ABC having cevian triangle DEF, then*

$$a^2\left(1 - \frac{BD}{DC} + \frac{BF}{FA}\right)\left(1 + \frac{CE}{EA} - \frac{CD}{DB}\right) + b^2\left(1 - \frac{BD}{DC} + \frac{BF}{FA}\right)\left(1 - \frac{CE}{EA} + \frac{CD}{DB}\right)$$
$$+ c^2\left(1 + \frac{BD}{DC} - \frac{BF}{FA}\right)\left(1 + \frac{CE}{EA} - \frac{CD}{DB}\right) = 0.$$

Conversely, any point P having ratio values that satisfy this equation will lie on the nine point circle of triangle ABC.

SOME PROBABILITY DISTRIBUTIONS AND THEIR ASSOCIATED STRUCTURES

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PART I

The well known binomial theorem for positive integral powers has had a long history of appearance in algebra texts. Rosenbach and Whitman [1] mention, for instance, how it was known to Omar Khayyam (c. 1100), extended by Isaac Newton (1642–1727) to fractional and negative values of n, and further worked on by Henrik Abel (1802–1829) who showed that the theorem holds for all values of n, including imaginary numbers.

The coefficients in the expansion of $(a+b)^n$ in the form of a table have had also a long and persistent appearance in algebra texts. The familiar triangular array of the binomial coefficients may be given as:

$$\begin{array}{ccccc} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \cdot \cdot \end{array} \qquad (1)$$

and is frequently mentioned as Pascal's triangle. It is known, however, that it appeared on the title page of an arithmetic published by Petrus Apianus in

1527, and some two centuries before in a work by a Chinese mathematician Chu Ski-kie [2].

In addition, the binomial coefficients for the positive integral values of n have also been used to form a binomial or Bernoulli distribution function for discrete probability processes [3]. It is the purpose of this paper to discuss an extension of this application to probability and attempt to relate the notion to a structure that is more general.

To begin, one must consider how the binomial coefficients may arise in some process when one is given a corresponding array of the positive integers. In fact, one wishes to discover a transformation, which may be signified as T_2, which will permit an equivalence relation to exist between the binomial coefficients and a corresponding array of the positive integers. In order to discover such a transformation one must first begin with the array of the ordered positive integers that corresponds to the array of the binomial coefficients. This may be written as:

$$\begin{array}{cccccccc} 1 & 2 & & & & & & \\ 2 & 3 & 4 & & & & & \\ 3 & 4 & 5 & 6 & & & & \\ 4 & 5 & 6 & 7 & 8 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n & \cdot & \cdot & \cdot & \cdot & 2n & & \end{array} \tag{2}$$

Now if these integers are not regarded as fixed and unique, but the results of some other, more fundamental, process, one may consider the process to be addition after permutations of the basis integers, the 1 and the 2. In this way the integers become what may be called consecutive sums, generated by addition after permutations, when the number of integers in a permutation equals the number of the row or n.

The n basis integers form what may be considered an inverse partition. That is, one begins with the fundamental elements, the 1 and the 2, and by the inverse partition process, plus all permutations of the inverse partitions, the consecutive sums are formed by addition.

One may now investigate how the array of the binomial coefficients is related to the array of the ordered positive integers (2). By inspection of the first row of the array of the integers one may note that the 1 and the 2 may be generated, when using one basis integer ($n=1$), in one and only one way. Hence an array listing the permutation sums for the row one of the integer array would begin with the numbers: 1 1. In row two of the integers, the numbers 2, 3, and 4, permit the partitions of 1, 1, then 2, 1 and 1, 2, and last of 2, 2. Hence, the array listing of the permutation sums for row two of the integers is: 1 2 1. Continuing, the row three integers permit the partitions: $3=1+1+1$, $4=1+1+2$, $5=2+2+1$, and $6=2+2+2$. Then the array listing of the permutations of these partitions becomes: 1 3 3 1. In the same way the row four of the integer array is associated with an array listing of: 1 4 6 4 1. Collecting the listings of the new array one has the following:

$$\begin{array}{ccccc} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{3}$$

At this point it can be noted that the array of the binomial coefficients is being generated. The nature of the equivalence relation or the transformation T_2 is revealed. Transformation T_2 generates the totality of the permutations of the inverse partitions of the consecutive sums into the basis integers.

Next, an extension of this process may be considered. Let the array of the positive integers be the following:

$$\begin{array}{ccccccc} 1 & 2 & 3 & & & & \\ 2 & 3 & 4 & 5 & 6 & & \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad 3n \end{array} \tag{4}$$

Now if one prepares the number of permutations of the basis integers or the 1, 2, and 3, which will generate by addition, the array of the bilinear ordered sums given above, the following array results:

$$\begin{array}{ccccccccc} 1 & 1 & 1 & & & & & & \\ 1 & 2 & 3 & 2 & 1 & & & & \\ 1 & 3 & 6 & 7 & 6 & 3 & 1 & & \\ 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{5}$$

This process appears to be quite general and if a given initial row of the integers is determined the remainder of the two arrays is quite dependent. For instance, a further example follows where the basis integers are 1, 2, 3, and 4. The sums array:

$$\begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & & & & & & & & & \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & & & & & & \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & & & \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad 4n \end{array} \tag{6}$$

The corresponding array for the number of the permutations of the inverse partitions has the corresponding entries:

$$\begin{array}{ccccccccccccc} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 1 & 3 & 6 & 10 & 12 & 12 & 10 & 6 & 3 & 1 \\ 1 & 4 & 10 & 20 & 31 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 \\ \cdots \end{array} \tag{7}$$

Here one observes that there are Pascal's triangles of the 1st kind, 2nd kind, $\cdots$, kth kind.

The interested reader can easily discover how other entries in these new triangular arrays are found if one remembers that post-row entries are generated by addition of pre-row entries above and left and takes care never to add more of the pre-row entries than the integral value of the number of the basis integers.

At this point it is of interest to attempt a general treatment of the ideas up to this point. If the array of the positive integers is given in a general way, such as:

$$\begin{array}{llll} 1 & 2 & \cdots k & \\ 2 & 3 & \cdots 2k & \\ 3 & 4 & \cdots 3k & \\ \cdots & & & k = 2, 3, 4, \cdots, m \\ n & n+1 & \cdots nk & n = 1, 2, 3, \cdots, r \end{array} \tag{8}$$

and if n and k are finite positive integral numbers, one has a system of sets that may be called C where $C = C_2, C_3, \cdots$, and C_m. Next if one considers T the set of transformations or $T_2, T_3, \cdots, T_m$, one may generate an isomorphic system of sets that may be called K where $K = K_2$ (this is the well known binomial series), $K_3, \cdots, K_m$. The operation may be stated as follows:

$$T(C) \to T_2C_2,\ T_3C_3, \cdots, T_mC_m \tag{9}$$

$$T_2C_2 \to K_2,\ T_3C_3 \to K_3, \cdots, T_mC_m \to K_m \tag{10}$$

The extended binomial series or $K = K_2, K_3, \cdots, K_m$, may take the following array in general form:

$$\begin{array}{lllll} 1 & 1 \cdots 1 & & & (k \text{ entries}) \\ 1 & 2 \cdots 2 & 1 & & (2k \text{ entries}) \\ 1 & 3 \cdots 3 & 1 & & (3k \text{ entries}) \\ \cdots & & & & \\ 1 & n \cdots n & 1 & & (nk \text{ entries}) \end{array} \tag{11}$$

At this time it will be of interest to summarize the connection between the two systems of sets, K and C.

Each entry in any row of a unique set of series of the system K is isomorphic with an entry from a unique set of the system C, the ordered bilinear positive integers. The connection between the entries of K and C become possible by the transformation T, where T signifies a process whereby each ordered integer is partitioned in all possible ways into the basis integers, using n of the basis integers, and then the total number of the permutations of all the possible partitions becomes the corresponding K entry. Briefly, one may note that the positive integers are projected into a statistical measure space by T.

In part two of this paper on probability distributions a discussion of an application will be given, a proposition on probability involving random variables will be given, and several novel features of the interlocking sets will be shown to have model value for the illustration and teaching of various topics of mathematics. (*To Be Continued*)

References

1. Rosenbach, Joseph B., Whitman, Edwin A., *College Algebra*, Ginn and Co., New York, 1939, p. 191.
2. Rosenbach, *op. cit.*, p. 193.
3. Hoel, Paul G., *Introduction to Mathematical Statistics*, John Wiley and Sons, Inc., New York, 1947, p. 39.

A SELF-DEFINING INFINITE SEQUENCE, WITH AN APPLICATION TO MARKOFF CHAINS AND PROBABILITY

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PART I

In this paper a type of infinite sequence will be considered, and some of its properties investigated. However, before defining the sequence for the general case, a very simple though interesting special case will be discussed. This will enable the reader to see how some of the methods used later apply to a numerical example.

I

The special sequence S is defined as follows:

a) A term of S is either the number 1 or the number 2.

b) S is composed of an infinite succession of finite segments $C_1, C_2, C_3, \cdots, C_i, \cdots$, so that

$$S = C_1C_2C_3 \cdots C_i \cdots.$$

c) The first two segments of S are:

$$C_1 = 211$$

$$C_2 = 2121.$$

d) Each segment C_{i+1} for $i \geqq 2$ is generated from the segment C_i according to the following rules:

(1) Replace each 1 in C_i by the set of numbers: 21

(2) Replace each 2 in C_i by the set of numbers: 211

For example,

$$C_3 = 2112121121$$
$$C_4 = 2112121211212112121211 21$$

so that the infinite sequence S starts as follows:

$$S = 211212121121211212121121212112121121212121 1 21 \cdots.$$

Let the number of 1's in the segment C_i be a_i, and the number of 2's in C_i be b_i. Since by definition d) there corresponds to each 1 in C_i just one 1 in C_{i+1}, and to each 2 in C_i there corresponds just two 1's in C_{i+1},

(1) $$a_{i+1} = a_i + 2b_i.$$

Also, since to each 1 in C_i there corresponds just one 2 in C_{i+1}, and similarly, to each 2 in C_i there corresponds just one 2 in C_{i+1},

(2) $$b_{i+1} = a_i + b_i.$$

Now let S_i be the partial sequence $C_1C_2C_3 \cdots C_i$. Also, let A_i be the total number of 1's in S_i and B_i be the total number of 2's in S_i. Then

(3) $$A_i = \sum_1^i a_k$$

(4) $$B_i = \sum_1^i b_k.$$

We now prove that

(5) $$A_{i+1} = A_i + 2B_i$$

(6) $$B_{i+1} = A_i + B_i.$$

Proof. For $k \geqq 2$, we have, from equations (1) and (2)

$$a_{k+1} = a_k + 2b_k$$
$$b_{k+1} = a_k + b_k.$$

Therefore

(i) $$\sum_2^i a_{k+1} = \sum_2^i a_k + 2\sum_2^i b_k$$

(ii) $$\sum_2^i b_{k+1} = \sum_2^i a_k + \sum_2^i b_k.$$

But

(iii) $$\sum_2^i a_{k+1} = \sum_1^{i+1} a_k - (a_1 + a_2) = A_{i+1} - (a_1 + a_2)$$

(iv) $$\sum_2^i b_{k+1} = \sum_1^{i+1} b_k - (b_1 + b_2) = B_{i+1} - (b_1 + b_2)$$

(v) $$\sum_{2}^{i} a_k = \sum_{1}^{i} a_k - a_1 = A_i - a_1$$

(vi) $$\sum_{2}^{i} b_k = \sum_{1}^{i} b_k - b_1 = B_i - b_1.$$

Therefore, from (i), (iii), (v) and (vi)

(vii) $$\begin{aligned} A_{i+1} - (a_1 + a_2) &= (A_i - a_1) + 2(B_i - b_1) \\ &= (A_i + 2B_i) - (a_1 + 2b_1). \end{aligned}$$

Similarly, from (ii), (iv), (v) and (vi)

(viii) $$\begin{aligned} B_{i+1} - (b_1 + b_2) &= (A_i - a_1) + (B_i - b_1) \\ &= (A_i + B_i) - (a_1 + b_1). \end{aligned}$$

But from definition (c)

$$\begin{aligned} a_1 + a_2 &= 4 = a_1 + 2b_1 \\ b_1 + b_2 &= 3 = a_1 + b_1. \end{aligned}$$

Therefore, from (vii) and (viii)

$$\begin{aligned} A_{i+1} &= A_i + 2B_i \\ B_{i+1} &= A_i + B_i \end{aligned}$$ Q.E.D.

Let N_i be the number of terms in S_i, and let t_k be the general term of the sequence. We then have:

a) The total number of terms in S_i equals

(7) $$N_i = A_i + B_i$$

b) The sum of the N_i terms of S_i equals

(8) $$\sum_{1}^{N_i} t_k = A_i + 2B_i.$$

From equations (7) and (8), the average of the first N_i terms of the infinite sequence S equals

$$\begin{aligned} \frac{\sum_{1}^{N_i} t_k}{N_i} &= \frac{A_i + 2B_i}{A_i + B_i} \\ &= 1 + \frac{B_i}{A_i + B_i} \\ &= 1 + \frac{1}{1 + \frac{A_i}{B_i}} \\ &= 1 + \frac{1}{1+} \frac{A_i}{B_i} \cdot \end{aligned}$$

But we can substitute for A_i and B_i in terms of A_{i-1} and B_{i-1} from equations (5) and (6) in which we change i to $(i-1)$, and we obtain:

$$\frac{\sum_1^{N_i} t_k}{N_i} = 1 + \frac{1}{1 +} \frac{A_{i-1} + 2B_{i-1}}{A_{i-1} + B_{i-1}}$$

$$= 1 + \frac{1}{1 +} 1 + \frac{B_{i-1}}{A_{i-1} + B_{i-1}}$$

$$= 1 + \frac{1}{2 +} \frac{1}{1 + \frac{A_{i-1}}{B_{i-1}}}$$

$$= 1 + \frac{1}{2 +} \frac{1}{1 +} \frac{A_{i-1}}{B_{i-1}} .$$

Again we can substitute from equations (5) and (6), and then reduce the continued fraction still more. But if we let i approach infinity, this substitution and reduction can be repeated an infinite number of times, and we therefore have:

$$\lim_{i\to\infty} \frac{\sum_1^{N_i} t_k}{N_i} = 1 + \frac{1}{2 +} \frac{1}{2 +} \frac{1}{2 +} \cdots.$$

However, N_i is a monotonically increasing function of i, so that as i approaches infinity, N_i does also. Therefore

$$\lim_{N_i\to\infty} \frac{\sum_1^{N_i} t_k}{N_i} = 1 + \frac{1}{2 +} \frac{1}{2 +} \frac{1}{2 +} \cdots. \tag{9}$$

Consider next the quadratic equation: $y^2+2y-1=0$, whose positive root is $(-1+\sqrt{2})$. The equation can be written as:

$$y(y + 2) = 1$$

$$y = \frac{1}{2 + y}$$

$$y = \frac{1}{2 +} \frac{1}{2 + y}$$

$$y = \frac{1}{2 +} \frac{1}{2 +} \frac{1}{2 +} \cdots \tag{10}$$

$$(-1 + \sqrt{2}) = \frac{1}{2 +} \frac{1}{2 +} \frac{1}{2 +} \cdots$$

$$\sqrt{2} = 1 + \frac{1}{2 +} \frac{1}{2 +} \frac{1}{2 +} \cdots.$$

Therefore, from equations (9) and (10), we have:

THEOREM. *The limit of the average of the first N_i terms of the sequence S, as the number of terms approaches infinity, equals $\sqrt{2}$.*

$$\lim_{N_i \to \infty} \frac{\sum_{1}^{N_i} t_k}{N_i} = \sqrt{2}.$$

I wish to thank Dr. I. A. Dodes and Mr. A. M. Glicksman of the Bronx High School of Science for their careful reading of earlier versions of this paper, and for the corrections and suggestions they made. I am also indebted to Professor B. P. Gill of The City College for the encouragement he gave me. My interest in the type of infinite sequence examined in this paper originated from a suggestion by my friend, Mr. Douglas Hofstadter, of a sequence very much like the one discussed in Part I.

POINT ALGEBRA

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Given the set of points $a, b, c, \cdots$ in a plane, we define addition and multiplication of these points as follows:

ADDITION. To find the sum, $a+b$, of two points a and b construct the line segment from a, through b, and terminating at c so that the distance between c and b equals the distance between b and a. Then c is the point $a+b$. (See Figure 1.)

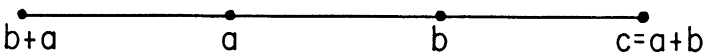


FIG. 1

MULTIPLICATION. To find the product, ab, of two points a and b first construct the line segment joining points a and b. Then, from the point a, construct a second line segment equal in length to the first and making an angle of 60°

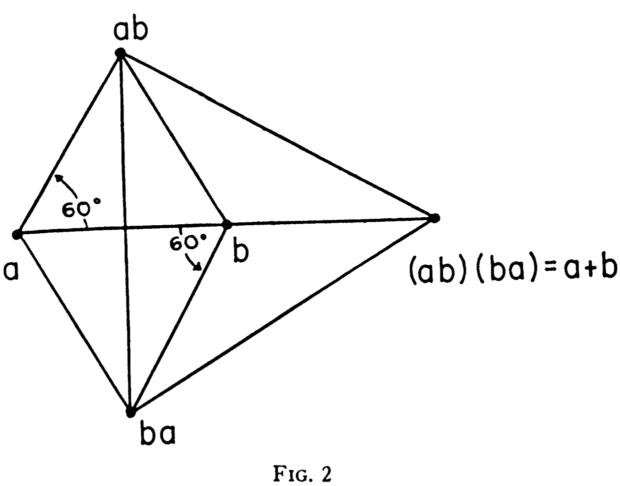


FIG. 2

with it. This second segment terminates at the point ab. (See Figure 2.)

From Figure 2 it is clear that

$$a(ba) = b, \tag{1}$$

and it is obvious that

$$aa = a. \tag{2}$$

Identity (3) is illustrated by Figure 3:

$$(ab)(cd) = (ac)(bd). \tag{3}$$

The geometrical verification of this is left as an exercise for the reader.

Finally, Figure 2 shows that

$$(ab)(ba) = a + b. \tag{4}$$

We are thus led to consider a system S with elements $a, b, c, \cdots$ that is closed under a binary operation $*$ satisfying the following postulates:

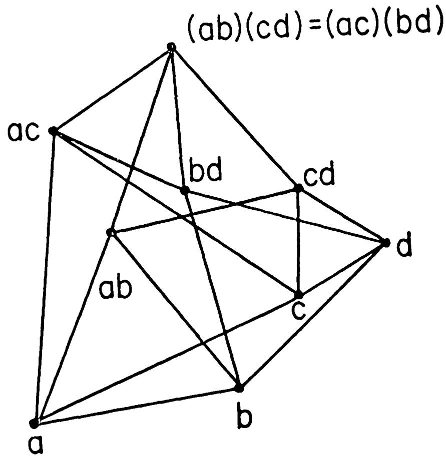


FIG. 3

1. $a * (b * a) = b$
2. $a * a = a$
3. $(a * b) * (c * d) = (a * c) * (b * d)$.

For convenience we write $a * b = ab$ and introduce the abbreviation $a + b = (ab)(ba)$. From these three postulates we may derive a large number of consequences, each of which, of course, may be illustrated geometrically.

THEOREM 1. *If a, b, c, d are in S we have*

(i) $a(cd) = (ac)(ad)$

(ii) $(ab)c = (ac)(bc)$

(iii) $a(ba) = (ab)a = b$

(iv) $a + a = a$

(v) $(a+b)+b = a$

(vi) $(a+b)(c+d) = ac+bd$

(vii) $(a+b)+(c+d) = (a+c)+(b+d)$

(viii) $a(c+d) = ac+ad$

(ix) $(a+b)c = ac+bc$

(x) $a(a+b) = a+ab$

(xi) $b(a+b)=ba+b$

(xii) $(a+b)a=a+ba$

(xiii) $(a+b)b=ab+b$

(xiv) $a+(c+d)=(a+c)+(a+d)$

(xv) $(a+b)+c=(a+c)+(b+c)$

(xvi) $(a+b)+a=a+(b+a)$

(xvii) $(a+b)(a+d)=a+bd$

(xviii) $(a+b)(c+b)=ac+b$

(xix) $(a+b)(b+a)=ab+ba$

(xx) $a(ba+a)=b+a$

(xxi) $(ba+a)(ba+b)=ba+ab$

(xxii) $a(ab)=ba+a$

(xxiii) $(ab)b=ba+b$

(xxiv) $[(ab)b]b=a+b$

(xxv) $ab(ba+a)=a$

(xxvi) $(b+a)+ab=ab(a+b)$

We will prove the first ten of these identities here, leaving the rest as exercises for the reader.

(i) By (3), $(aa)(cd)=(ac)(ad)$. But, by (2), $aa=a$.
(ii) By (3), $(ab)(cc)=(ac)(bc)$. But, by (2), $cc=c$.
(iii) By (i) $a(ba)=(ab)(aa)$. But $aa=a$ and $a(ba)=b$.
(iv) By the definition of $a+b$ we have $a+a=(aa)(aa)$. But $aa=a$.
(v) By the definition of $a+b$ we have

$$(a+b)+b=(ab)(ba)+b=\{[(ab)(ba)]b\}\{b[(ab)(ba)]\}.$$

But

$$\begin{aligned}\{[(ab)(ba)]b\}\{b[(ab)(ba)]\} &= \{[(ab)b][(ba)b]\}\{[b(ab)][b(ba)]\} \text{ by (ii) and (i)}\\ &= \{[(ab)b]a\}\{a[b(ba)]\} \text{ by (iii)}\\ &= \{[(ab)a][ba]\}\{[ab][a(ba)]\} \text{ by (ii) and (i)}\\ &= [b(ba)][(ab)b] \text{ by (iii)}\\ &= [b(ab)][(ba)b] \text{ by (3)}\\ &= aa \text{ by (iii)}\\ &= a \text{ by (2).}\end{aligned}$$

(vi)

$$\begin{aligned}(a+b)(c+d) &= [(ab)(ba)][(cd)(dc)]\\ &= [(ab)(cd)][(ba)(dc)] \text{ by (3)}\\ &= [(ac)(bd)][(bd)(ac)] \text{ by (3)}\\ &= ac+bd \text{ by the definition of } a+b.\end{aligned}$$

(vii)

$$\begin{aligned}(a+b)+(c+d) &= (ab)(ba)+(cd)(dc)\\ &= (ab+cd)(ba+dc) \text{ by (vi)}\\ &= [(a+c)(b+d)][(b+d)(a+c)] \text{ by (vi)}\\ &= (a+c)+(b+d) \text{ by the definition of } a+b.\end{aligned}$$

(viii) By (vi), $(a+a)(c+d)=ac+ad$. But, by (iv), $a+a=a$. (See Figure 4 for an illustration of this identity.)
(ix) By (vi), $(a+b)(c+c)=ac+bc$. But, by (iv), $c+c=c$.
(x) By (viii), $a(a+b)=aa+ab$. But $aa=a$.

It is convenient to make the further definition that $b=c-a$ if and only if $a+b=c$. It is clear that the point $c-a$ bisects the interval (a, c) in our geometrical interpretation.

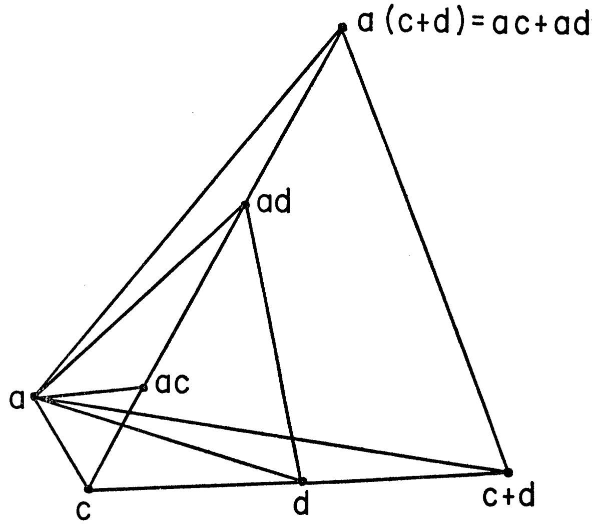


FIG. 4

THEOREM 2. *If a, b, c, d are in S we have*

(i) $a-a=a$

(ii) $a+(c-a)=c$

(iii) $a-c=c-a$

(iv) $a+(c-a)=a+(a-c)=c$

(v) $(a-b)+(c-d)=(a+c)-(b+d)$

(vi) $(a-b)-(c-d)=(a-c)-(b-d)$

(vii) $ab-cd=(a-c)(b-d)$

(viii) $ab-ba=a-b$

(ix) $c-ab=(c-a)(c-b)$

(x) $ab-c=(a-c)(b-c)$

(xi) $a-(b+c)=(a-b)+(a-c)$

(xii) $(b+c)-a=(b-a)+(c-d)$

(xiii) $a+(b-c)=(a+b)-(a+c)$

(xiv) $(b-c)+a=(b+a)-(c+a)$

(xv) $a+(a-c)=a-(a+c)=c$

(xvi) $(a-c)+a=a-(c+a)$

(xvii) $c(a-b)=ca-cb$

(xviii) $(a-b)c=ac-bc$

(xix) $a(a-b)=a-ab$

(xx) $b(a-b)=ba-b$

(xxi) $(a-b)a=a-ba$

(xxii) $(a-b)b=ab-b$

We will prove (v) and (vi) leaving the rest as exercises for the reader.

(v) Let $b+e=a$ and $d+f=c$. Then $e=a-b, f=c-d$, and $(b+e)+(d+f)=a+c$. But, by (vii) of Theorem 1, $(b+e)+(d+f)=(b+d)+(e+f)$. Thus $e+f=(a+c)-(b+d)$ by definition of subtraction. Hence $(a-b)+(c-d)=(a+c)-(b+d)$.

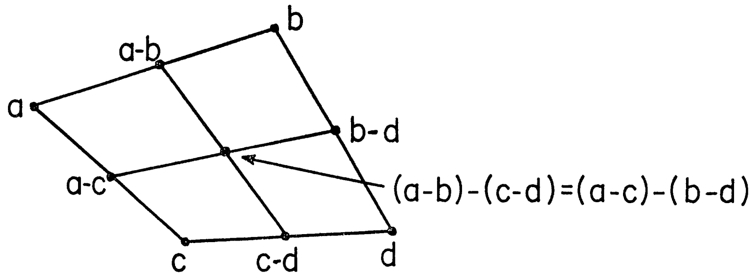


FIG. 5

(vi) With the same notation as above we have $e-f=(a-b)-(c-d)$ and $(b+e)-(d+f)=a-c$. But, by (v) of Theorem 2, $(b+e)-(d+f)=(b-d)+(e-f)$. Hence $(b-d)+(e-f)=a-c$, and by definition of subtraction, $e-f=(a-c)-(b-d)$. (See Figure 5 for an illustration of this identity.)

We now make the further definition that $c=a:b$ if and only if $c+a=b+c$. It is clear from Figure 6 that the point $a:b$ is a point of trisection of the interval (a, b) in our geometrical interpretation.

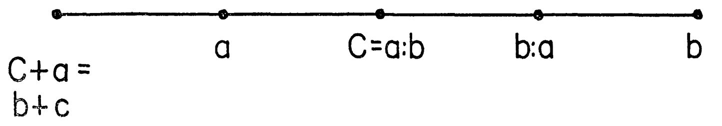


FIG. 6

THEOREM 3. *If a, b, c, d are in S we have*

(i)	$a:a=a$	(v)	$a:b-c:d=(a-c):(b-d)$
(ii)	$(b+a):(a+b)=a$	(vi)	$(a:b):(c:d)=(a:c):(b:d)$
(iii)	$(a:b)(c:d)=(ac):(bd)$	(vii)	$b:a+a:b=a$
(iv)	$a:b+c:d=(a+c):(b+d)$	(viii)	$a:(a+b)=b:a$

(ix) $(a-c):b=(c-b):a=(b-a):c$

We will prove (i), (ii) and (iii), leaving the rest as exercises for the reader.

(i) Since $a+a=a+a$, we have $a:a=a$ by the definition of $a:b$.

(ii) By (xvi) of Theorem 1, $a+(b+a)=(a+b)+a$ and $a=(b+a):(a+b)$ by the definition of $a:b$.

(iii) Let $e+a=b+e$ and $f+c=d+f$. Then $e=a:b$, $f=c:d$, and $(e+a)(f+c)=(b+e)(d+f)$. But $(e+a)(f+c)=ef+ac$ and $(b+e)(d+f)=bd+ef$ by (vi) of Theorem 1. Thus $ef=(a:b)(c:d)=(ac):(bd)$ by the definition of $:$.

In view of the geometrical interpretations of $a-b$ and $a:b$, we see that identity (ix) of Theorem 3 proves the following geometric theorem:

The medians of a triangle meet at a point which is two thirds of the distance from each vertex to the midpoint of the opposite side.

We may define $a \circ b=(a:b)(b:a)$ and obtain the following theorem, the proof of which is left to the reader.

THEOREM 4. *If a, b, c, d are in S we have*

(i)	$(a \circ b) \circ (c \circ d)=(a \circ c) \circ (b \circ d)$	(viii)	$a \circ b+b \circ a=ba$
(ii)	$(a \circ b)(c \circ d)=ac \circ bd$	(ix)	$ba \circ (a+b)=b$
(iii)	$a \circ b+c \circ d=(a+c) \circ (b+d)$	(x)	$a \circ b=(b \circ a)b=a(b \circ a)$
(iv)	$a \circ b-c \circ d=(a-c) \circ (b-d)$	(xi)	$a \circ (b \circ a)=a:b$
(v)	$(a \circ b):(c \circ d)=(a:c) \circ (b:d)$	(xii)	$a \circ (a+b)=(a \circ b)b$
(vi)	$(a \circ b)(b \circ a)=b$	(xiii)	$(a \circ b) \circ b=(a \circ b)(b:a)=(b:a)b$
(vii)	$(a \circ b) \circ (b \circ a)=b:a$	(xiv)	$(a \circ b) \circ (a:b)=(a:b) \circ (b:a)$

(xv) $a \circ a = a$

TEACHING OF MATHEMATICS

EDITED BY ROTHWELL STEPHENS, Knox College

This department is devoted to the teaching of mathematics. Thus, articles of methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, as a teacher, are interested, or questions which you would like others to discuss, should be sent to Rothwell Stephens, Mathematics Department, Knox College, Galesburg, Illinois.

THE THEORY OF NUMBERS IN AMERICA TODAY

WILLIAM EDWARD CHRISTILLES, St. Mary's University, Texas

Throughout history the theory of numbers has occupied a distinguished and colorful position in the world of mathematics. Almost every century has seen new and fascinating discoveries made in this field, many times in almost totally unexpected areas. Most of the great mathematicians have probed into its mysteries at sometime in their career and contributed in its development.

Why then in twentieth century America should number theory be forced to occupy an almost secondary position of study in our colleges and universities? In many universities in our country no more than one three-hour course is offered on the undergraduate or graduate level. Many graduate schools fail to offer either the M.S. or Ph.D. in this area of mathematics. Should we allow ourselves to become so engrossed in modern trends in mathematics that we find ourselves in a lagging position with respect to other nations, particularly European, in this field of mathematics?

Admittedly number theory has contributed very little as an applied science, although some few applications have been found. Applications alone, however, should not be the determining factor in deciding the value of mathematics. Mathematics has an esthetic value, which, in itself justifies both its existence and study. Many mathematicians, like artists, enjoy the study of mathematics for itself rather than for its scientific value; and this is by no means an inappropriate attitude. A great number of the outstanding contributions both in mathematics and science have resulted from the researches of just such men. We must continue to make a place in our system of education for this type of dedicated man.

The theory of numbers is perhaps the only field of mathematics which so closely adheres to the experimental and scientific approaches for its development. Most branches of pure mathematics depend upon deductive forms of reason in their development, particularly with the current trend toward axiomatic methods in mathematics. Yet the research worker in number theory must rely to a great extent upon trial and error and his intuition. Ultimately, of course, he will not be satisfied until a rigid mathematical proof of the matter under consideration has been formulated. Although the author has the highest regard for axiomatic mathematics, he does not feel that number theory, due to its very nature, is particularly well suited for that type of development.

Another mistake that many authors make is in trying to unify number theory into a systematic study and to some extent they succeed; however, in so doing they destroy much of the beauty of the subject itself. One of the finest works in the literature which strongly substantiates this view is Dickson's History of the Theory of Numbers. In his three volume work Dickson emphasized the point of view of the research worker rather than writing for an organized classroom presentation. In this manner, he relates the true nature of creative development in number theory. The implication shown in his work is that discoveries in this branch of mathematics are spotty rather than systematic and that clear and organized proofs of many of its elaborate theorems follow the initially cumbersome and isolated first type proofs.

Perhaps the most unusual characteristic of number theory is its attraction for both the professional and the amateur mathematician. This is probably due to the very basic nature of the problems in number theory. Moreover, although many of the problems in this field require a great deal of maturity in order to solve them, they are still fundamentally very easy to interpret. Further, the great variety of methods employed in solving problems in this field seems to be almost inexhaustible, a fact that seems to be particularly appealing to the creative individual who likes to devise his own means of solving problems. It is the one field in mathematics in which the amateurs have made many noticeable discoveries; and, by so doing, they have gained the admiration and respect of the professionals. This has resulted in a close feeling of unity between the two groups which is seldom felt in other areas of mathematics.

Surely then both from the historical standpoint and the fact that the study of number theory has resulted in so much enjoyment for those who have studied it, this field of mathematics is worthy of a more thorough treatment in our colleges and universities. There are some schools such as the University of Chicago and, more recently, the University of Colorado which have emphasized this field of study; but the small colleges and universities throughout our country seem to be neglecting its study. Evidently they feel that only those courses designed to meet the immediate practical needs of their students are worthy of consideration. Yet to fully educate a man we must satisfy his total intellectual interests. He should be educated in a manner which will develop in him a deep esthetic feeling toward his field. It is the author's view that this feeling is most adequately acquired in mathematics in such areas as geometry and the theory of numbers and that their study should be emphasized rather than neglected.

In conclusion, the author would like to state that he does not agree with the many mathematicians who say that number theory will always be a totally "pure" branch of mathematics, void of "practical" applications. Number theory, as has been pointed out, is more closely related to the physical sciences in its development than any other of the major fields of mathematics. It is as basic to mathematics as Einstein's concepts were to the physical universe. For these reasons, the author feels that in the future, as both the fields of science and mathematics progress, number theory will become a principle tool in explaining the basic physical concepts of one or more of the physical or behavioral sciences.

A SIMPLE PROOF OF DESCARTES' RULE OF SIGNS

P. V. KRISHNAIAH, Andhra University, India

The usual (Algebraic) proof of the well known Descartes' rule of signs depends upon the following

LEMMA. *If $P(x)$ is a real polynomial and $a>0$ then the number of changes of sign in (the sequence of non-zero coefficients in) $(x-a)P(x)$ is greater than that of those in $P(x)$.*

The "proof" of this lemma, which appears in literature, is in fact a diagrammatic persuasion, whose rigorous presentation is rather lengthy. I now prove this by induction on the number of changes of sign in $P(x)$.

If all the coefficients in $P(x)$ are of the same sign, then since the first and the last terms in $(x-a)P(x)$ have opposite signs, there is at least one change of sign in $(x-a)P(x)$.

Suppose that when $P(x)$ has r changes of sign, $(x-a)P(x)$ has at least $(r+1)$ changes of sign.

Let $P(x)$ be a polynomial having $(r+1)$ changes of sign and suppose the $(r+1)$th change of sign occurs just after a_sx^s. We can write $P(x)$ as $Q(x)+R(x)$ where $Q(x)$ has r changes of sign, and

$$Q(x) \equiv a_0 + a_1x + \cdots + a_sx^s, \qquad a_s \neq 0,$$

$$R(x) \equiv a_{s+t}x^{s+t} + \cdots + a_nx^n, \quad t \geqq 1, \quad a_sa_{s+t} < 0 < a_{s+t}a_n.$$

Now

$$(x - a)P(x) \equiv (x - a)Q(x) + (x - a)R(x).$$

Let $a>0$. If $t>1$, then $(x-a)Q(x)$ has at least $(r+1)$ changes of sign by the induction hypothesis and $(x-a)R(x)$ has at least one since $-aa_{s+t}$ and a_n are of opposite sign.

If $t=1$, the last term of $(x-a)Q(x)$ and the first term of $(x-a)R(x)$ merge. Since a_s and $-aa_{s+1}$ have the same sign, the number of changes of sign in $(x-a)Q(x)-aa_{s+1}x^{s+1}$, being the same as that of those in $(x-a)Q(x)$, is at least $(r+1)$.

Since a_s-aa_{s+1}, which is the coefficient of x^{s+1} in $(x-a)P(x)$, and a_n are of opposite sign, there is at least one change of sign in $(x-a)P(x)$, after the term x^{s+1}. Thus in any case $(x-a)P(x)$ has at least $(r+2)$ changes of sign. Hence the lemma.

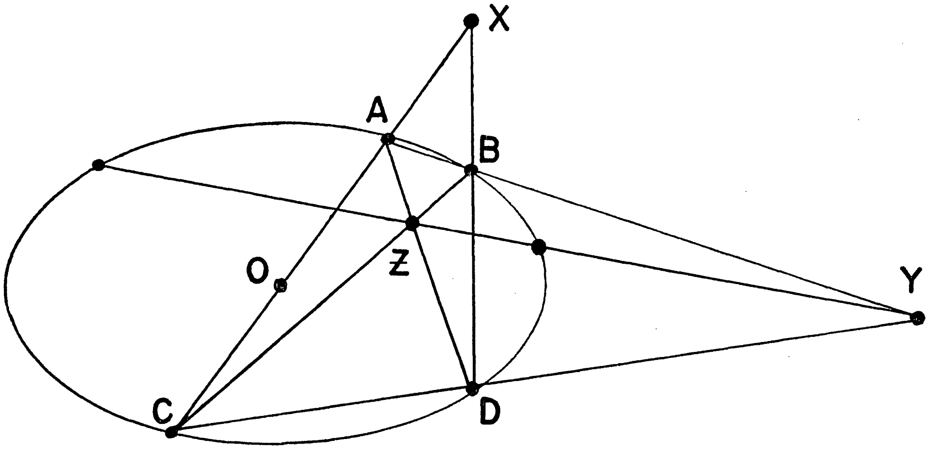
ON A COMPLETE QUADRANGLE INSCRIBED IN A CONIC

JAYME MACHADO CARDOSO, University of Paraná, Brazil

All conics are here supposed nondegenerate.

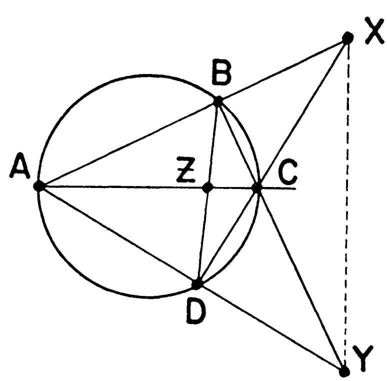
PROPOSITION 1. *If a complete quadrangle inscribed in a conic with center has one side passing through the center of the conic, the line which passes through the diagonal points not belonging to this side has the conjugate direction of this same side.*

Proof. Let $ABCD$ be the vertices of a complete quadrangle inscribed in a conic with center and with one side, AC, passing through the center of the conic. Its diagonal points are X, Y, Z. The pole of the line ZY is the point X. As AC passes through the center of the conic, the pole of AC belongs to the point at infinity of the line ZY, i.e., the tangents at A and C are parallel to YZ.



COROLLARY. *If a complete quadrangle inscribed in a circle has a side as diameter, this side is perpendicular to the line joining the diagonal points which do not belong to this side.*

This corollary can be easily established independently from the Proposition 1. For, in the triangle AXY the lines BY and DX are perpendicular to the sides AX and AY. Then, CA is also perpendicular to XY.



Also trivial is the proof of the following:

PROPOSITION 2. *If a complete quadrangle inscribed in a parabola has one of its vertices at infinity in the direction of the axes, the line joining two diagonal points is parallel to the tangent at the vertex of the quadrangle which is on the side parallel to the axes passing through the other diagonal point.*

COMMENTS ON PAPERS AND BOOKS

EDITED BY HOLBROOK M. MACNEILLE, Case Institute of Technology

This department will present comments on papers published in the MATHEMATICS MAGAZINE, lists of new books, and reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent to Holbrook M. MacNeille, Department of Mathematics, Case Institute of Technology, Cleveland 6, Ohio.

SUPPLEMENT TO AN ARTICLE ENTITLED "ON THE USE OF THE EQUIVALENCE SYMBOL AND PARENTHESES SYMBOLS IN ASSOCIATIVE DISTRIBUTIVE ALGEBRA"

H. S. VANDIVER,* The University of Texas

In my paper published in the September-October 1959 issue of MATHEMATICS MAGAZINE, under the above-mentioned title, I stated in footnote 4, "In algebra, at least, one of the most valuable tools is *homomorphism*. This seems to be a concept which is not employed in logic." In April of 1960 Stephen A. Kiss, after receiving a reprint of my article, wrote me concerning this statement as follows: "While the first part of this statement is correct, the second part seems to be in error, since the true propositions of a logical system form an ideal according to the Tarski-Stone theory and, therefore, the true and the false propositions merely form two of the residue-classes corresponding to this ideal." In view of this, I naturally wished to discuss in MATHEMATICS MAGAZINE the statement of mine which Mr. Kiss mentions and refer my readers to some published article or book where a concise account of the above-mentioned theory could be found. I had no knowledge of the Tarski-Stone theory myself, and I was unable to locate any article or book by either Tarski or Stone on subjects pertaining to logic which explained in full the nature of it. In fact it seems probable that no such an account existed in the literature at the time my article was written. I then wrote Kiss for his assistance in locating pertinent literature, and in May of 1961 I was happy to receive from him a booklet he had just written entitled "An Introduction to Algebraic Logic," April 1961.† This booklet consists of 34 pages (with four pages, "Addenda and Corrigenda," added in November 1961). In this work Kiss explains what he called the Tarski-Stone theory, from the beginning, so that the meaning of the statement made in his original letter to me is fully explained in a set of definitions, theorems and corollaries obtained mainly by combining ideas‡ from two articles written by Alfred Tarski and one written by M. H. Stone. These articles are as follows:

Alfred Tarski, "Grundzüge des Systemenkalkül, Erster Teil," Fundamenta Mathematicae, **25** (1935), 503–26.

* The work on this paper was sponsored by the National Science Foundation, Grant 19665.

† Published by Stephen A. Kiss, 3 Laurel Lane, Westport, Connecticut.

‡ In addition, Kiss proves a number of independent results in algebraic logic in his booklet. He also notes that Tarski himself, in the English translation of *Logic, Semantics, Metamathematics*, page 350, recognized a connection with some of his own work with that of the 1936 paper of Stone. A review of the book by Kiss referred to above appears in Math. Reviews, Vol. 23, No. 5A, 1962, Review A3080. A concise description of the nature of the Tarski-Stone theory is included. However, as Kiss noted in a letter to me, said review makes no mention of his independent results.

Alfred Tarski, "Grundzüge des Systemenkalkül, Zweiter Teil," Fundamenta Mathematicae, **26** (1936), 283–301.

M. H. Stone, "The Theory of Representation for Boolean Algebras," Trans. Amer. Math. Soc., **40** (1936), 37–111.

The two articles by Tarski also appear on pages 342–383 in his book entitled *Logic, Semantics, Metamathematics*, translated by J. H. Woodger, Oxford University Press, 1956.

BOOK REVIEWS

Techniques du Calcul Matriciel (*Techniques of Matrix Calculus*). By D. Pham, Dunod, Paris, 1962.

This book which treats precisely the different techniques of matrix calculus is written for mathematicians as well as for physicists and engineers.

The properties of matrices are explained from the elementary point of view; the exposition leads progressively to the definition of various canonical forms as well as to applications to systems of linear equations and to the solution of linear differential equations.

The proofs are complete and rigorous enough to satisfy the pure mathematician and simple enough to be understood by the applied mathematician. The completely solved numerical examples illustrate the theorems and the precise remarks made on the methods to be employed will be of great service to those intending to solve numerical problems on the machine.

The table of contents include the following chapter headings: 1. Elements of Algebraic Forms; 2. Matrices; 3. Elementary Transformation of Matrices; 4. Similarity of Square Matrices, Canonical Forms; 5. Euclidean Spaces, Hermitian Spaces and Congruences; 6. Functions of Matrices; 7. Systems of Linear Equations; 8. Characteristic Values and Vectors; 9. Systems of Linear Differential Equations with Constant Coefficients; 10. Systems of Linear Differential Equations with Variable Coefficients. Appendix I. Systems of linear equations. Methods of Kacmarg and Cimmino. Appendix II. Determination of characteristic values and vectors of a matrix. Methods of Givens and Lanczos.

The book is a valuable addition to the literature of matrix calculus.

SOUREN BABIKIAN
Los Angeles City College

Arithmêtique Gênêrale. By A. Doneddu, Dunod, Paris, 1962.

This book is written to meet the requirements introduced in the new mathematics programs of the French Baccalaureate system of education. This new program includes such basic concepts of modern mathematics as sets, correspondence, axiomatic method, systems of numeration to any base, concepts of group, ring and ideals.

The students starting college and university work in mathematics find themselves in difficulty by the sudden and abrupt change to the language and symbolism of modern abstract mathematics. The object of this book is to enable such students in their preparatory mathematical education to familiarize themselves with the notations and ideas of modern mathematics.

In the first chapter the new vocabulary and symbolism are introduced and basic ideas of sets are defined and explained. The rest of the book is divided into four principal parts.

Part one is on construction of the set N of natural numbers by the use of Peano axioms. All the fundamental operations of arithmetic are explained in terms of sets and axiomatic method. In this section the three chapters on congruences, rules of divisibility and prime numbers are remarkably well written.

Part two is about the construction of rational numbers and the b-nary numbers concluding with a noteworthy chapter on the theory of numerical approximation.

Part three consists of the construction of the set of real numbers and the measure of magnitudes using the axiom of Archimedes.

Part four is on construction of the set k of relative numbers and their operations, concluding with the final chapter on exponential and logarithmic functions.

This is a unique book on the theory of arithmetic. There is not a comparable book in English on theoretical arithmetic at this level. It is an excellent book for students intending to become future mathematicians and also for teachers of mathematics on the secondary level.

SOUREN BABIKIAN
Los Angeles City College

Elements of Linear Spaces. By A. R. Amir-Moéz and A. L. Fass, Edwards Brothers, Inc., Ann Arbor, 1961, vii+149 pp.

This book approaches the study of the elements of linear spaces through geometry. The first of three parts presents vectors, transformations, and matrices and their use in the study of three-dimensional Euclidean space. In the second part the ideas previously introduced are generalized for the study of both real and complex n-dimensional spaces. A quadratic form in a unitary space E_n is defined, and an application is made to the classification of conic sections and quadric surfaces. The third part introduces the study of modern abstract algebra and the general notion of vector spaces. The last chapter presents some recent results on singular values and estimates of proper values of matrices.

The authors have succeeded remarkably well in developing two-and three-dimensional geometry by means of vectors and matrices, whereas many writers of elementary books on linear algebra are content with a few applications to geometry. The generalizations to n-dimensional geometry and to abstract algebra are also here skillfully introduced.

The material has been carefully organized and clearly presented. The illustrative examples are well-chosen and abundant, and the exercises range from routine to challenging.

The pages of the book are large, $8\frac{1}{2}''$ by $11''$. The illustrative figures are displayed quite well, and the print and the arrangement of the material on the page make the book attractive in appearance and easy to read.

There are, of course, a few flaws. In Part 1 the reader is left to infer that *the* *space* means three-dimensional space. In the opinion of the reviewer it would have been better to include the answers to some of the exercises. The book would also benefit by the inclusion of a bibliography, even a brief one. It is hoped that the many typographical errors in the reviewer's copy of the book have been corrected in subsequent printings.

JANET McDONALD
Vassar College

Basic Concepts in Modern Mathematics. By John E. Hafstrom. Addison-Wesley, Reading, Mass., 1961. x+195 pp.

This is a text designed for a one-semester terminal course in basic mathematical concepts for freshmen who have had a year of algebra and a year of geometry. The natural number system is defined in Chapter 2 by a reasonably simple set of nine axioms, the extension to the set of all integers is made in standard fashion in Chapter 7, while the intevening chapters prepare the student for this extension by a discussion of set operators, functions, relations, partitions and groups. The material in these first seven chapters constitutes a cohesive, well written text which should clear up most of the misconceptions acquired in high school algebra and give the students an acquaintanceship with some of the basic concepts of modern mathematics.

There are two additional chapters. Chapter 8 involves the extension of the integers to the rationals and is quite straightforward except for one blunder. The author defines the rationals as the set of all equivalent ordered pairs of integers (with second member non-zero), defines addition and multiplication, and claims that an isomorphism exists between the rationals and the fractions which he defines to be these ordered pairs of integers. Chapter 9 discusses the Cauchy sequence technique for extending the rationals to the reals and proves that the reals constitute a field. This chapter is marred by the author's frequent use of infinite decimals which are nowhere defined.

Chapter 9 is of a much higher order of difficulty than the rest of the text and is probably unintelligible to the students. The author contends that "the inclusion of these chapters [8 and 9] makes the book appropriate for an introductory course in modern mathematics designed for students who have completed a year of calculus. Such a course could serve as a bridge between the traditional freshman-sophomore mathematics sequence and a course in modern abstract algebra based on, say, Birkhoff and MacLane's *A Survey of Modern Algebra* or a similar text." This is sheer wishful thinking. Any student who has had a year of calculus and feels he can take more mathematics is ready for stronger meat than this book offers.

L. J. GREEN
Case Institute of Technology

BOOKS RECEIVED FOR REVIEW

Introduction to Differentiable Manifolds. By Serge Lang, Wiley, New York, 1962, x+126 pages, $7.00.

Diophantine Geometry. By Serge Lang, Interscience, New York, 1962, x+170 pages, $7.45.

Fourier Analysis on Groups. By Walter Rudin, Interscience, New York, 1962, ix+285 pages, $9.50.

Propositional Calculus. By P. H. Nidditch, Macmillan, New York, 1963, viii+83 pages, $1.25 (paper).

Truth-Functional Logic. By J. A. Faris, Macmillan, New York, 1963, vi+122 pages, $1.25 (paper).

The Mathematical Theory of Optimal Processes. By L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, Wiley, New York, 1962, viii+360 pages, $11.95.

Local Rings. By Masayoshi Nagata, Wiley, New York, 1962, xiii+234 pages, $11.00.

Principles of Abstract Algebra. By Richard W. Ball, Holt, Rinehart and Winston, New York, 1963, ix+290 pages, $6.00.

Matrix Iterative Analysis. By Richard S. Varga, Prentice-Hall, Englewood Cliffs, N. J., 1963, xiii+322 pages, $7.50.

Matrix Algebra for Social Scientists. By Paul Horst, Holt, Rinehart and Winston, New York, 1963, xxi+517 pages, $10.00.

Matrix Methods for Engineering. By Louis A. Pipes, Prentice-Hall, Englewood Cliffs, N. J., 1963, xiii+427 pages, $9.75.

Projective and Euclidean Geometry. By W. T. Fishback, Wiley, New York, 1962, ix+244 pages, $7.50.

Geometry. By Frank M. Morgan and Jane Zartman, Houghton Mifflin, Boston, 1963, xii+615 pages, $4.96.

Elementary Concepts of Mathematics, Second Edition. By Burton W. Jones, Macmillan, 1963, xvii+350 pages, $5.50.

The Laplace Transform. By Earl D. Rainville, Macmillan, 1963, vi+106 pages, $2.50 (paper).

Integral and Differential Calculus, An Intuitive Approach. By Hans Sagan, Wiley, New York, 1962, xiii+329 pages, $5.95.

University Calculus. By Howard E. Taylor and Thomas L. Wade, Wiley, New York, 1962, xxi+765 pages, $9.95.

The Calculus. By William L. Schaaf, Doubleday, New York, 1963, viii+436 pages, $1.95 (paper).

Elements of Algebra. By Calvin A. Rogers, Wiley, New York, 1962, xiii+320 pages, $5.95.

Modern Algebra and Trigonometry. By Mary P. Dolciani, Simon L. Berman and William Wooton, Houghton Mifflin, Boston, 1963, xi+658 pages, $5.20.

Analytic Geometry, Second Edition. By Paul K. Rees, Prentice-Hall, Englewood Cliffs, N. J., 1963, xi+275 pages, $6.95.

Introductory College Mathematics. By Patrick Shanahan, Prentice-Hall, Englewood Cliffs, N. J., 1963, ix+340 pages, $6.50.

New Understanding in Arithmetic. By James R. Smart, Allyn & Bacon, Boston, 1963, ix+268 pages, $5.95.

Mathematics for Everyday Use. By William L. Schaaf, Doubleday, New York, 1963, x+349 pages, $1.45 (paper).

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.

PROPOSALS

516. *Proposed by Maxey Brooke, Sweeny, Texas.*

Six men, Adams, Brown, Jones, McCall, Smith, and Williams are sitting at equal intervals around a circular table.

That is, five of them are sitting. The sixth is slumped in that so-dead position corpses assume. One of the men at the table is his murderer. We know the following facts:

1. Jones sits to the left of the uncle of the man just across the table from him.
2. The murdered man had no relatives.
3. Adams asks McCall, who is sitting next to him, for a cigarette.
4. The murderer does not sit next to the uncle or the nephew, but the murdered man sits between them.
5. The man on Smith's right, who is not Jones, sits next to the murderer.
6. Adams sits directly across the table from Williams who is smoking nervously.

Who was killed by whom?

517. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let F and d be the focus and directrix of a parabola. If M and N are any two points on the parabola and M', N' are their respective projections on d, show that

$$\frac{\text{Area } FMN}{\text{Area } N'M'\text{MN}} = \text{Constant.}$$

518. *Proposed by Murray S. Klamkin, State University of New York at Buffalo.*

Show that an integer is determined uniquely from a knowledge of the product of all its divisors.

519. *Proposed by Michael J. Pascual, Watervliet Arsenal, New York.*

A point is chosen at random on a line segment of length l. What is the probability that an isosceles triangle can be formed by using either segment as the base and the remaining segment as the bisector of the base angle?

520. *Proposed by L. S. Shively, Ball State Teachers College.*

A prodigy gave 1.94608974115866 as the value of one of the roots, to fifteen digits, of the equation

$$x^9 - 9x^7 + 27x^5 - 30x^3 + 9x + 1 = 0.$$

A certain thirteen of the fifteen digits are correct and the other two are incorrect. Which ones are incorrect, and what is the value of this root, correct to fifteen digits?

521. *Proposed by Leo Moser, University of Alberta.*

Prove that

$$\sum_{i=1}^{2n-1} 2^{i-1}\binom{4n-2}{2i}$$

is a perfect square for $n=1, 2, 3, \cdots$.

522. *Proposed by Daniel P. Shine, Villa Madonna College, Kentucky.*

In how many ways can n objects, not all different, be arranged in a circle:
a. if neither reflections nor rotations are counted?
b. if reflections but not rotations are counted?
c. if rotations but not reflections are counted?
d. if both rotations and reflections are counted?

SOLUTIONS

Late Solutions

470, 477, 481, 482, 483, 484, 485, 487. *Josef Andersson, Vaxholm, Sweden.*

489, 490. *Josef Andersson, Vaxholm, Sweden, and Thomas R. Hamrick, San Quentin, California.*

491, 493. *Josef Andersson, Vaxholm, Sweden.*

A Particularly Peculiar Pythagorean Perimeter

495. [November 1962] *Proposed by Maxey Brooke, Sweeny, Texas.*

"I just surveyed a peculiar field," said an engineer, "it is the smallest right triangle with integral sides whose perimeter is a cube." What are the dimensions of the field?

I. *Solution by Monte Dernham, San Francisco, California.*

The perimeter of a Pythagorean triangle is given by

$$(1) \qquad 2kr(r+s), \qquad r > s, \qquad (r, s) = 1, \quad r \text{ or } s \text{ even.}$$

Thus the cube sought is that of an even number which is not a power of 2, $r+s$ being odd. The smallest such cube is $2^3 \cdot 3^3 = 6^3 = 216$. To avoid the introduction of a prime factor other than 2 or 3, we are restricted to $r=2$, $s=1$. Then $k=216/(2^2\cdot 3)=18$. Applying the familiar formula for Pythagorean triples, and considering the "smallest right triangle . . . " to be the one with the shortest perimeter, we conclude that the sides of the surveyed field are

$$(2) \qquad 54, \quad 72, \quad 90; \qquad \text{area, } 1944.$$

As there are instances where a triangle with longer perimeter than another has a smaller area, we proceed to show that the foregoing result remains unchanged when the area instead of the perimeter is regarded as the criterion for the "size" of a triangle. Since the area of a Pythagorean triangle whose perimeter $\geqq 20^3$ is clearly >1944, and since the cubes of odd numbers and of powers of 2 are inadmissible, it suffices to examine only four sets of integral right triangles, those whose perimeters are, respectively,

$$(3) \qquad 10^3 = 2^3 \cdot 5^3; \qquad 12^3 = 2^6 \cdot 3^3; \qquad 14^3 = 2^3 \cdot 7^3; \quad \text{and} \quad 18^3 = 2^3 \cdot 3^6.$$

The labor of identifying and testing these is shortened by the following considerations. First, the value of r is restricted to powers of 2, $<2^6$, since for $r \geqq 2^6$ the area would clearly exceed $64^2/2 = 2048$, and if r were not a power of 2 then *either* at least one of its factors would be a factor of s, contradicting (1), *or* else $r(r+s)$ would contain two or more distinct odd factors, contradicting (3). Secondly, for the reason just mentioned, $r+s$, being odd, is restricted in each instance to a power of 3, of 5, or of 7. It will then be found that, aside from (2), the perimeters of but six Pythagorean triangles constitute cubes of numbers <20:

r	s	$r+s$	k	Sides			Perimeter	Area
4	1	5	25	375	200	425	$1000 = 10^3$	37,500
8	1	9	12	756	192	780	$1728 = 12^3$	72,576
16	11	27	2	270	704	754	$1728 = 12^3$	95,040
2	1	3	144	432	576	720	$1728 = 12^3$	124,416
4	3	7	49	343	1176	1225	$2744 = 14^3$	201,684
2	1	3	486	1458	1944	2430	$5832 = 18^3$	1,417,176

As seen, in no instance is the area $\leqq 1944$, which confirms (2) as the correct solution regardless of the precise meaning of "smallness" in reference to a triangle.

II. *Solution by J. A. H. Hunter, Toronto, Ontario, Canada.*

Let us assume that the proposer intended the smallest primitive pythagorean triangle. Solving the diophantine equation $x^2+xy=4z^3$, with 4-parameter solution $x=4b^3cd^2$, $y=(a^3c-4b^3d)cd$, $z=abcd$, for a primitive triangle we must have $c=d=1$. This solution then becomes: $x=4b^3$, $y=a^3-4b^3$, $z=ab$, with $4b^3<a^3<8b^3$. The smallest primitive pythagorean triangle, with perimeter a cube, then is generated by $a=5$, $b=3$, giving sides 11953, 11375, and 3672.

Also solved for primitive pythagorean triangles by J. W. Blundon, Memorial University of Newfoundland; George Diderick, University of Wisconsin; Francis L. Miksa, Aurora, Illinois; Sam Sesskin, Hempstead, New York; and the proposer.

Those solving for composite pythagorean triangles include *Leonard Bertain, St. Mary's College, California; Sister Marie Blanche, The Immaculata, Washing-*

ton, D. C.; Daniel I. A. Cohen, Brooklyn, New York; Sister M. Coleman, Marywood College, Pennsylvania; Jay Gottesfeld, AVCO, Wilmington, Massachusetts; Thomas R. Hamrick, San Quentin, California; Murray S. Klamkin, State University of New York at Buffalo; Sidney Kravitz, Dover, New Jersey; Gilbert Labelle, Université de Montréal; Herbert R. Leifer, Pittsburgh, Pennsylvania; C. N. Mills, Sioux Falls College, South Dakota; J. W. Milsom, Texas A and I College; E. E. Morrison, University of Aberdeen, Scotland; Sam Newman, Atlantic City, New Jersey; Jerry L. Pietenpol, Columbia University; R. Ransom, Montreal, Canada; Alfred Shaeftler, New South Wales, Australia; David L. Silverman, Beverly Hills, California; Ralph Vawter, St. Mary's College, California; William C. Weber, Derry, Pennsylvania; and Hazel S. Wilson, Jacksonville University, Florida. One incorrect solution was received.

Mixed Cryptarithm

496. [November 1962] *Proposed by C. W. Trigg, Los Angeles City College.*

Solve this mixed cryptarithm in which each letter uniquely represents a digit in the decimal system:

$$\begin{array}{cccccc} &&&E&R&E\\ &&&*&*&*\\ \hline &&*&*&2&1\\ &L&O&N&G&\\ &*&*&*&&\\ \hline *&R&*&*&*&* \end{array}$$

Solution by Edward L. Spitznagel, Jr., University of Chicago.

The only pairs of numbers from 0 to 9 whose products have 1 for the units digit are (1, 1), (3, 7), and (9, 9). We therefore have four choices for E: 1, 3, 7, and 9.

$E=1$ implies that $**21=ERE$ and so is out.

$E=3$ and $E=7$ are also out, for both imply that $R=0$, so that $1=N$ and $0=G$.

For $E=9$, we have:

$$\begin{array}{cccccc} &&&9&6&9\\ &&&*&*&9\\ \hline &&8&7&2&1\\ &L&O&N&G&\\ &*&*&*&&\\ \hline *&6&*&*&*&1 \end{array}\;;\quad \text{then:}\quad \begin{array}{cccccc} &&&9&6&9\\ &&&1&*&9\\ \hline &&8&7&2&1\\ &L&O&N&G&\\ &9&6&9&&\\ \hline *&6&*&*&*&1 \end{array}\;;\quad \text{and finally:}\quad \begin{array}{cccccc} &&&9&6&9\\ &&&1&6&9\\ \hline &&8&7&2&1\\ &5&8&1&4&\\ &9&6&9&&\\ \hline 1&6&3&7&6&1 \end{array}$$

Also solved by Merrill Barneby, University of North Dakota; Leonard Bertain, St. Mary's College, California; Joseph B. Bohac, St. Louis, Missouri; Maxey

Brooke, Sweeny, Texas; Daniel I. A. Cohen, Brooklyn, New York; Sister M. Coleman, Marywood College, Pennsylvania; Monte Dernham, San Francisco, California; George Diderick, University of Wisconsin; Lillian K. Fishbein, Public School 188, Brooklyn, New York; Harry M. Gehman, State University of New York at Buffalo; Jay Gottesfeld, AVCO, Wilmington, Massachusetts; Thomas R. Hamrick, San Quentin, California; J. A. H. Hunter, Toronto, Ontario, Canada; Sidney Kravitz, Dover, New Jersey; Gilbert Labelle, Université de Montréal; Herbert R. Leifer, Pittsburgh, Pennsylvania; John W. Milsom, Texas A and I College; Jerry L. Pietenpol, Columbia University; Alfred Shaeftler, New South Wales, Australia; David L. Silverman, Beverly Hills, California; Donval R. Simpson, University of Alaska; H. L. Skala, Mundelein College, Illinois; Hazel S. Wilson, Jacksonville College, Florida; Robert F. Winter, New York University; Brother Louis Zirkel, Archbishop Mulloy High School, Jamaica, New York; and the proposer.

Equivalent Binomial Sums

497. [November 1962] *Proposed by Murray S. Klamkin, AVCO, Wilmington, Massachusetts.*

Show that

$$4\sum_{r=0}^{n} r^3\binom{n}{r}^p = 6n\sum_{r=0}^{n} r^2\binom{n}{r}^p - n^3\sum_{r=0}^{n}\binom{n}{r}^p.$$

Solution by Francis D. Parker, University of Alaska.

This problem is equivalent to showing that

$$\sum_{r=0}^{n}\left[4r^3 - 6nr^2 + n^3\right]\binom{n}{r}^p = 0.$$

Let $f(r) = 4r^3 - 6nr^2 + n^3$ and $g(r) = \binom{n}{r}^p$.

It follows easily that $f(r) = -f(n-r)$ and that $g(r) = g(n-r)$. From these results the conclusion is immediate.

Also solved by L. Carlitz, Duke University; Jerry L. Pietenpol, Columbia University; and the proposer.

A Property of Multiplicative Functions

498. [November 1962] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If m and n are integers and δ, D are their g.c.d. and l.c.m. respectively, and $d(n)$ denotes the number of divisors of n, $\phi(n)$ being the Euler function, prove that:

(1) $$d(m)d(n) = d(\delta)d(D)$$

(2) $$\phi(m)\phi(n) = \phi(\delta)\phi(D)$$

Solution by L. Carlitz, Duke University.

The result is a special case of the following theorem. Let $f(n)$ be an arbitrary factorable function, that is

$$f(mn) = f(m)f(n)$$

for all m, n such that $(m, n)=1$. Then

$$(*) \qquad f(m)f(n) = f(\delta)f(D),$$

where $\delta=(m, n)$ and $D=[m, n]$, the greatest common divisor and the least common multiple, respectively.

The proof of (*) is immediate. If

$$m = \Pi p^r, \qquad n = \Pi p^s,$$

then

$$\delta = \Pi p^{r'}, \qquad D = \Pi^{s'},$$

where $r'=\min(r, s)$, $s'=\max(r, s)$. Since $r'+s'=r+s$, we have

$$f(\delta)f(D) = \Pi p^{r'+s'} = \Pi p^{r+s} = f(m)f(n).$$

Also solved by Stephen R. Cavior, Duke University; Daniel I. A. Cohen, Brooklyn, New York; George Diderick, University of Wisconsin; Murray S. Klamkin, State University of New York at Buffalo; David A. Klarner, Humboldt State College, California; Gilbert Labelle, Université de Montréal; Jerry L. Pietenpol, Columbia University; Robert Prielipp, University of Wisconsin; Sam Sesskin, Hempstead, New York; David L. Silverman, Beverly Hills, California; Irene Williams, Converse College, South Carolina; and the proposer.

Random Points

499. [November 1962] *Proposed by N. C. Perry, Auburn University, Alabama.*

Prove that if n points are chosen at random on a circle, the probability that they all lie on the same semi-circle is $n/2^{n-1}$.

I. *Solution by Jerry L. Pietenpol, Columbia University.*

If the smallest arc containing the first $(n-1)$ points is ϕ, then the probability that θ is the smallest arc containing n points is $\phi/2\pi$ for $\theta=\phi$, and $1/\pi \, d\theta$ for $\phi<\theta<\pi$.

Thus let $w_n(\theta)d\theta$ be the probability distribution of θ, for angles up to π, for n points. Then

$$w_n(\theta) = \frac{\theta}{2\pi} w_{n-1}(\theta) + \frac{1}{\pi}\int_0^\theta w_n(\phi)d\phi,$$

$$w_2(\theta) = \frac{1}{\pi},$$

so that

$$w_n(\theta) = \frac{n(n-1)}{(2\pi)^{n-1}} \theta^{n-2}.$$

The probability that n points lie on a semicircle is

$$\int_0^\pi w_n(\theta)d\theta = n/2^{n-1}.$$

II. *Solution by W. W. Funkenbusch, Michigan College of Mining and Technology.*

Let us prove the contention by induction.

Part (1) $P_1 = 1/2^0 = 1$, which checks with obvious fact.

Part (2) Assume true if $n = k$, i.e. that $P_k = k/2^k - 1$. Now k points and their opposites divide the circumference of the circle into $2k$ arcs, each of which has the same expectation as to length. If k points are on "a" same semi-circle and an additional point be spotted at random on the circle, if it falls within $k+1$ of the $2k$ arcs, the $k+1$ points are on "a" same semi-circle. Hence

$$P_{k+1} = \frac{k+1}{2k} P_k = \frac{k+1}{2k} \cdot \frac{k}{2^{k-1}} = \frac{k+1}{2^k}$$

and the induction is complete.

Also solved by Daniel I. A. Cohen, Brooklyn, New York; David L. Silverman, Beverly Hills, California; and the proposer.

Isovolume Transformation

500. [November 1962] *Proposed by Joseph Hammer, University of Sidney, Australia.*

If x_1', x_2', x_3' are the images of the vectors x_1, x_2, x_3 of 3-dimensional space under the linear transformation

$$\begin{aligned} x_1' &= a_1x_1 + a_2x_2 + a_3x_3 \\ x_2' &= b_1x_1 + b_2x_2 + b_3x_3 \\ x_3' &= c_1x_1 + c_2x_2 + c_3x_3 \end{aligned}$$

where the values of $a_1, a_2, \cdots, c_3$ are the integers from 1 to 9 taken in some order, which ordering will make the volume of the parallelopiped defined by x_1', x_2', x_3' equal to the volume of the one defined by x_1, x_2, x_3?

Solution by Dale Woods, Northeast Missouri State Teachers College.

If the two volumes are to be equal then the Jacobian of the transformation must be ± 1. Since

$$\begin{vmatrix} 1 & 7 & 4 \\ 9 & 8 & 5 \\ 6 & 3 & 2 \end{vmatrix} = 1$$

then one solution is given by $a_1 = 1$, $a_2 = 7$, $a_3 = 4$, $b_1 = 9$, $b_2 = 8$, $b_3 = 5$, $c_1 = 6$, $c_2 = 3$, $c_3 = 2$.

Also solved by Murray S. Klamkin, State University of New York at Buffalo; Jerry L. Pietenpol, Columbia University; and the proposer.

Stewart's Theorem

501. [November 1962] *Proposed by Paul D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

In the usual triangle ABC, with corresponding opposite sides a, b, c, denote the median, internal bisector, and symmedian issued from A by t_m, t_a, and t_s respectively. Let t be the join of A to the point P on BC which bisects the perimeter. Show that:

(1) $t_s = Ht_m/(2M-H)$ where M, H are the arithmetic and harmonic means of b and c

(2) $t_a = H \cos 1/2A$

(3) $$t^2 = s^2\left(1 - \frac{r}{R}\sec^2\frac{1}{2}A\right)$$

where s is the semi-perimeter, r and R the inradius and circumradius of the given triangle.

Solution by the proposer.

Stewart's theorem in the appropriate form is

$$ad^2 = nb^2 + mc^2 - amn, \; m + n = a.$$

$$t_a{:}m/n = b/c, \; m = ab/(b+c), \; n = ac/(b+c)$$

$$t_s{:}m/n = b^2/c^2, \; m = ab^2/(b^2+c^2) \; n = ac^2/(b^2+c^2)$$

$$t_m{:}m = n = \tfrac{1}{2}a \qquad t{:}m = s - b, \; n = s - c$$

From the above find

$$t_m = \tfrac{1}{2}\sqrt{2(b^2+c^2) - a^2}$$

$$t_s = bc\sqrt{2(b^2+c^2) - a^2}/(b^2+c^2).$$

Whence

$$t_s = 2bct_m/(b^2+c^2) = Ht_m/(2M - H).$$

$$t^2 = s^2\left(1 - \frac{r}{R}\sec^2 \tfrac{1}{2}A\right).$$

Also solved by Jay Gottesfeld, AVCO, Wilmington, Massachusetts.

Comment on Problem 338

338. [March and November 1958] *Comment by Murray S. Klamkin, State University of New York at Buffalo.*

The method quoted from Brand's Advanced Calculus doesn't work for the case here. The function $R \times R/r^3$ is obviously zero and so $f(R) \neq \text{curl } G$.

In the meanwhile, I have discovered the problem elsewhere, i.e.,

1. Vector Analysis, H. B. Phillips, p. 109.

$$G(R) = -\cos\theta \nabla\phi + \nabla\psi$$

where θ and ϕ are spherical coordinates and ψ arbitrary.

2. Vector and Tensor Analysis, H. Lass, p. 118.

$$G(R) = \frac{x}{r(y^2+z^2)}(-zj + yk) + \nabla\psi.$$

3. Vector Methods, D. E. Rutherford, p. 100.

$$G(R) = \frac{yzi}{3r}\left\{\frac{z^2-y^2}{(x^2+y^2)(x^2+z^2)}\right\} + \frac{zxj}{3r}\left\{\frac{x^2-z^2}{(y^2+z^2)(y^2+x^2)}\right\} + \frac{xyk}{3r}\left\{\frac{y^2-x^2}{(z^2+x^2)(z^2+y^2)}\right\} + \nabla\psi$$

which is a symmetrical solution.

Comment on Problem 440

440. [March and November 1961, November 1962] *Comment by Alan Sutcliffe, Knottingley, Yorkshire, England.*

The expression for N given by the proposer is not correct, as the example in the solution clearly shows. Although it does give the correct value of N for some values of R/r, for example when R/r is an odd integer, it underestimates the value of N in many cases.

The expression for N given in the solution is the correct one for the number of small circles contained entirely in the large one, tangency being included: the proposer's suggestion that it includes also the partly contained circles is not correct. This is confirmed by the example given in the solutions for the case where $R/r = 2\sqrt{3}+1$. In this case:

the proposer's expression gives $N=7$,
the solver's expression gives $N=13$,
the number of wholly contained circles is 13,
the number of wholly and partly contained circles is 31.

For proof of the solver's expression in general, reference may be made to the solution.

Comment on Problem 470

470. [January and September 1962] *Comment by Henry W. Gould, West Virginia University.*

I think the solution below is probably one of the simplest which we can find. It uses nothing such as any strong theorem of Gauss or any hypergeometric functions, and uses really only the simple fact that the $(n+1)$st difference of a polynomial of degree n is zero.

Here is the solution again:

$$\sum_{k=0}^{n}(-1)^k\frac{(2n+1-k)!(n+1)!}{(2n+2)!(k+1)!(n-k)!(n+1-k)!}$$
$$=\frac{n!}{(2n+2)!}\sum_{k=0}^{n}(-1)^k\binom{n+1}{n-k}\binom{2n+1-k}{n}$$
$$=(-1)^n\frac{n!}{(2n+2)!}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{n+1+k}{n}$$
$$=(-1)^n\frac{n!}{(2n+2)!}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\binom{n+1+k}{n}+\frac{1}{(n+2)!}.$$

But it is clear that $\binom{x}{n}$ is a polynomial of degree n in x, so that the summation here is the $n+1$ difference of a polynomial of degree n and hence must be zero. Thus the value of the proposed sum is merely $1/(n+2)!$ as desired to show.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 313. In how many distinct patterns can 9 congruent squares—3 red, 3 white, and 3 blue—be arranged in a square array so that all three colors occur in every column and row? [*Submitted by C. W. Trigg.*]

Q 314. It follows by symmetry that the line joining the centers of two congruent, parallel, intersecting ellipses bisects the common chord. Show that the result holds if the ellipses are no longer congruent but similar. [*Submitted by M. S. Klamkin.*]

Q 315. Prove that if each side of a quadrilateral, expressed in inches, is a factor of the sum of the other three sides, then at least two of the sides are equal. [*Submitted by D. L. Silverman.*]

Q 316. The last non-zero digit of the factorial of each positive integer greater than 1 is even. [*Submitted by C. W. Trigg.*]

Q 317. A determinant whose elements are either 0 or 1 has a value ± 1. What is the maximum and minimum number of ones? [*Submitted by M. S. Klamkin.*]

(Answers on page 156)

References

1. Fenton Stancliff, "Nine-Digit Determinants," Scripta Mathematica, **19** (1953), 278.
2. Vern Hoggatt and F. W. Saunders, "Maximum Value of a Determinant," American Mathematical Monthly, **62** (1955), 257.
3. C. W. Trigg and D. C. B. Marsh, "Probability that a Determinant Be Even," American Mathematical Monthly, **70** (1963), 93.
4. Charles W. Trigg, "Vanishing Nine-Digit Determinants," School Science and Mathematics, **62** (1962), 330–331.
5. Charles W. Trigg, "Unit-Valued Nine-Digit Determinants," Nabla (Bull. Malayan Math. Soc.), **8** (1961), 185–186.
6. Charles W. Trigg, "Third Order Determinants Invariant under Element Interchange," The Australian Mathematics Teacher, **18** (1962), 40–41.

Answers

A 313. Write the colors in order diagonally on a cylindrical surface,

$$\begin{matrix} R & W & B & r & w \\ b & R & W & B & r \\ w & b & R & W & B \end{matrix}$$

The lower case letters indicate how the diagonals will fall to complete three vertical columns. By progressing around the cylinder, three distinct patterns are obtained:

RWB	*WBR*	*BRW*
BRW	*RWB*	*WBR*
WBR	*BRW*	*RWB*

The last two could have been obtained from the first by cyclic permutation of the letters. The three clearly are different since the central squares have different colors. All other apparently different arrangements go into one of these *three* by rotation.

A 314. The result is obviously true for two intersecting circles. Consequently it is true for two similar parallel ellipses by orthogonal projection.

A 315. If each side divides the sum of the other three, then each side divides the perimeter P. If no two sides are equal then $P/4 <$ the longest side $<P/2$ so that the longest side equals $P/3$. But then the sum of the four sides is at most $P/3+P/4+P/5+P/6<P$. Hence at least two sides are equal.

A 316. The factorial of each positive integer, $1<n<5$, contains at least one even factor. The factorial of every larger positive integer contains as factors at least two even integers for every integer ending in five. The product of one of these even integers by five introduces a terminating zero. Of the remaining factors, at least one is even, thus insuring that the last non-zero digit of $n!$ is even.

A 317. Obviously the minimum number is n. The maximum number is n^2-n+1 which occurs in the determinant $|A_{rs}|$ where $A_{rs}=1-\delta_{1,r-s}$ where as usual the $\delta_{m,n}=0$ for $m\neq n$, and $\delta_{m,m}=1$.

(Quickies on page 206)

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